ABSTRACT HARMONIC ANALYSIS AND WAVELETS IN $\mathbb{R}^n$

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Abstract. Necessary and sufficient conditions are given for an integrable integer-valued function to be the multiplicity function of a generalized multiresolution analysis in $L^2(\mathbb{R}^n)$, and also for it to be the multiplicity function of an expansive matrix dilation wavelet in $L^2(\mathbb{R}^n)$. For possible multiplicity functions, an explicit construction is given for the GMRA with a tight frame, and also for an associated MSF wavelet.

Introduction

In the past ten years, wavelets have emerged as an increasingly powerful tool of harmonic analysis on $\mathbb{R}^n$. Wavelets themselves have been studied primarily using tools of classical Fourier analysis. In this paper we show some of the ways in which more general techniques of abstract harmonic analysis can be exploited to give new insights into wavelets. In particular, we use the multiplicity function of Stone [S] and Mackey [M] as a tool for producing and analyzing wavelets.

Many examples of wavelets have been produced using the related concept of a multiresolution analysis. However, well-known examples due to Journé and others ([DL]) show that not all wavelets have an associated MRA. Indeed, Dai, Larson and Speegle ([DLS]) have shown that single wavelets can be produced in $L^2(\mathbb{R}^n)$ for any dilation δ_A (where A is an expansive matrix) and any dimension n, yet a result in [BCMO] proves that MRA wavelets are possible only if $|\det A| = 2$. In [BMM] we introduced the concept of generalized multiresolution analyses, and showed that together with a condition on the multiplicity function we call the consistency equation, they give a necessary and sufficient condition for the existence of a wavelet.

In Section 1 of this paper we further explore the properties of generalized multiresolution analyses. We show that every GMRA whose multiplicity function is finite a.e. has a tight frame, so that this apparently more general structure is the same as the generalized frame multiresolution analyses of Papadakis ([P]), and includes the frame multiresolution analyses of Benedetto ([B]). We then go on to give a necessary and sufficient condition for an integrable integer-valued function on the n-torus to produce a GMRA in $L^2(\mathbb{R}^n)$. Our proof gives an explicit method for constructing both the GMRA and its frame.

In Section 2, we combine our work on GMRA's with the results in [BMM] to give two conditions on an integrable integer-valued function that are necessary and sufficient for it to produce a wavelet. For dilation by 2 wavelets in $L^2(\mathbb{R}^1)$ the multiplicity function has been shown ([W]) to equal the dimension function of

Auscher ([A]). Because of this, our work in Section 2 generalizes some recent results of Rzeszotnik and Speegle on dilation by 2 wavelets in $L^2(\mathbb{R}^1)$ ([RS]).

Many of the techniques of this paper can be employed in studying multiwavelets and wavelets on a general Hilbert space (see [BMM]). However, in this paper we will only consider single wavelets on $L^2(\mathbb{R}^n)$ and its subspaces.

Generalized Multiresolution Analyses

We view the lattice $\mathbb{Z}^n$ as a group Γ of unitary operators on $L^2(\mathbb{R}^n)$ acting by $\gamma_n f(x) = f(x+n)$. We parametrize $\widehat{\Gamma} = \mathbb{R}/\Gamma$ by $[-\pi, \pi)^n$ with addition mod 2π. For any $n \times n$ integer matrix A, all of whose eigenvalues have absolute value greater than 1, we let δ_A be the unitary operator on $L^2(\mathbb{R}^n)$ given by

$$[\delta_A(f)](x) = |\det(A)|^{\frac{1}{2}} f(Ax).$$

We note, for later use, that a simple calculation shows that the actions of Γ and δ_A are interrelated by the formula $\delta_A^{-1}\gamma_n\delta_A = \gamma_{An}$. In this context we recall the following definition, introduced in [BMM]:

DEFINITION. By a *generalized multiresolution analysis*(GMRA) we shall mean a collection $\{V_j\}_{-\infty}^{\infty}$ of closed subspaces of $L^2(\mathbb{R}^n)$ that satisfy:

(1) $V_j \subseteq V_{j+1}$ for all j.
(2) $\delta_A(V_j) = V_{j+1}$ for all j.
(3) $\cup V_j$ is dense in $L^2(\mathbb{R}^n)$ and $\cap V_j = \{0\}$.
(4) V_0 is invariant under the action of Γ.

If the closure of $\cup V_j$ is a proper subspace of $L^2(\mathbb{R}^n)$ we say that the collection $\{V_j\}$ is a *subspace GMRA*.

The definition of a GMRA differs from that of an MRA in condition (4): an MRA replaces (4) with the stronger requirement that V_0 contains a scaling vector whose translates form an orthonormal basis for V_0. Although a GMRA need not have a scaling function, by studying the action of Γ on the invariant subspace V_0, we will be able to produce a tight frame to take its place.

Let $\{V_j\}$ be any GMRA, and write ρ for the unitary representation of the group Γ given by its action on V_0. By the spectral theorem for commutative groups, ρ is uniquely determined by a projection valued measure on $\widehat{\Gamma}$. This projection valued measure is in turn uniquely determined by an ordinary measure class on $\widehat{\Gamma}$ and a multiplicity function $m : \widehat{\Gamma} \mapsto \{\infty, 0, 1, 2, \cdots\}$. (See, e.g. [He], [M], [Ha], [S].) We show in [BMM] that the measure class associated with a GMRA in $\mathbb{R}^n$ must be absolutely continuous with respect to Lebesgue measure. Thus the multiplicity function, which roughly measures how many times each character in $\widehat{\Gamma}$ occurs in ρ, completely characterizes the representation for a GMRA in $L^2(\mathbb{R}^n)$. If the GMRA is an MRA, translates by Γ of the single function ϕ give an orthonormal basis for V_0. Thus in this case, the representation ρ of Γ on V_0 is equivalent to the regular representation of Γ, which acts by translation on $l^2(\Gamma)$. The regular representation is known to (weakly) contain every character exactly once, so in the MRA case we have $m \equiv 1$. We will see that many other multiplicity functions are possible for GMRA's. We will restrict our attention to the case where m is finite almost everywhere.

To use the information the multiplicity function provides about ρ, we form the direct sum $L^2(S_1) \oplus L^2(S_2) \oplus \cdots$, where $S_j = \{x \in [-\pi, \pi)^n : m(x) \geq j\}$. Write $\tilde{\rho}$ for the representation of Γ on $L^2(S_1) \oplus L^2(S_2) \oplus \cdots$ given by $\tilde{\rho}_\gamma(f_1, f_2, \cdots) = (e^{i<\gamma,\cdot>} f_1, e^{i<\gamma,\cdot>} f_2, \cdots)$. The properties of m guarantee the existence of a unitary map $J : V_0 \mapsto \oplus_{j=1}^{\infty} L^2(S_j)$ which intertwines the actions of Γ on the two spaces, so that $J\rho_\gamma f = \tilde{\rho}_\gamma Jf$ for all $f \in V_0$. (See [Co] for more details.)

We can use the map J to produce a tight frame for the V_0 space of any GMRA:

DEFINITION. A set of vectors $\{\phi_n : n \in \mathbb{Z}\}$ is called a *frame* for a subspace $V \subset L^2(\mathbb{R}^n)$ if there exist positive constants C_1 and C_2 such that for all $f \in V$,

$$C_1 \|f\|^2 \leq \sum_{n \in \mathbb{Z}} | < f, \phi_n > |^2 \leq C_2 \|f\|^2.$$

It is called a *tight frame* if $C_1 = C_2$.

Let $\phi_j = J^{-1}(\chi_{S_j})$ (where χ_{S_j} stands for the element of $\oplus_{i=1}^{\infty} L^2(S_i)$ whose only nonzero component is χ_{S_j} in $L^2(S_j)$); if $S_j = \emptyset$ we take $\phi_j = 0$. We have the following Lemma, the $L^2(R^1)$ case of which appears in ([Co]):

LEMMA 1.1. *If $\{V_j\}$ is a GMRA whose multiplicity function is finite almost everywhere, then $\{\phi_j(\cdot + \gamma) : j \geq 1, \gamma \in \Gamma\}$ is a tight frame for V_0.*

Proof. Let $f \in V_0$. For any $g \in L^2[-\pi, \pi)^n$ and $\gamma \in \Gamma$, let $c_\gamma(g)$ be the γ Fourier coefficient of g. Then

$$\begin{aligned}
\sum_{\gamma \in \Gamma} \sum_{j=1}^{\infty} | < f, \rho_\gamma \phi_j > |^2 &= \sum_{\gamma \in \Gamma} \sum_{j=1}^{\infty} | < J(f), \tilde{\rho}_\gamma \chi_{S_j} > |^2 \\
&= \sum_{j=1}^{\infty} \sum_{\gamma \in \Gamma} |c_\gamma(J(f)\chi_{S_j})|^2 \\
&= \sum_{j=1}^{\infty} \|J(f)\chi_{S_j}\|^2 \\
&= \|J(f)\|^2 \\
&= \|f\|^2.
\end{aligned}$$

This tight frame plays the role in a GMRA of the orthonormal basis given by the scaling function of an MRA. Because we cannot in general write down an explicit formula for the operator J, we cannot always explicitly construct the tight frame. However, we will later see cases in which explicit construction is possible. In any case, this frame satisfies many of the familiar properties of the scaling function. In particular, we will need the following:

LEMMA 1.2. $\sum_{\gamma \in \Gamma} \hat{\phi}_i(x + 2\pi\gamma)\overline{\hat{\phi}_j(x + 2\pi\gamma)} = \begin{cases} \chi_{S_i}(x) \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}$ *for almost all $x \in [-\pi, \pi)^n$.*

Proof. Both sides can be considered as periodic functions on $\mathbb{R}^n$, so that it will suffice to show they have the same Fourier coefficients. Accordingly, let c_γ be the

γ^{th} Fourier coefficient of the $L^2(\Gamma)$ inner product $< \hat{\phi}_i(x+2\pi\cdot), \hat{\phi}_j(x+2\pi\cdot) >$, considered as a function on $[-\pi,\pi]^n$. Then

$$\begin{aligned}
c_\gamma &= \int_{[-\pi,\pi]^n} e^{-i<\gamma,x>} \sum_{\gamma'\in\Gamma} \hat{\phi}_i(x+2\pi\gamma')\overline{\hat{\phi}_j(x+2\pi\gamma')}dx \\
&= \sum_{\gamma'\in\Gamma} \int_{[-\pi,\pi]^n+2\pi\gamma'} e^{-i<\gamma,x>} \hat{\phi}_i(x)\overline{\hat{\phi}_j(x)}dx \\
&=< \hat{\phi}_i, e^{i<\gamma,\cdot>}\hat{\phi}_j > \\
&=< \phi_i, \rho_\gamma \phi_j > \\
&=< \chi_{S_i}, \tilde{\rho}_\gamma \chi_{S_j} > \\
&= \begin{cases} c_\gamma(\chi_{S_i}) \text{ if } i=j \\ \quad 0 \text{ if } i \neq j. \end{cases}
\end{aligned}$$

To produce m and guarantee a frame, we have only used the action of Γ on V_0. Now, we will analyze m further by using δ_A to conjugate ρ up to V_1. Let W_0 be the orthogonal complement of V_0 in V_1.

LEMMA 1.3. *If m is the multiplicity function associated with a GMRA in $L^2(\mathbb{R}^n)$ then m must satisfy*

$$\sum_{y\in[-\pi,\pi)^n : A^T y \equiv x \bmod 2\pi} m(y) \geq m(x) \tag{1}$$

for almost all $y \in [-\pi,\pi)^n$.

Proof. First we claim that V_1 is invariant under the action of Γ on $L^2(\mathbb{R}^n)$. For, if $f = \delta_A g$ is any element in V_1, with $g \in V_0$, then $\gamma_n f = \gamma_n \delta_A g = \delta_A(\delta_A^{-1}\gamma_n\delta_A)g = \delta_A \gamma_{An} g \in V_1$. We write ρ^1 for the action of Γ restricted to V_1. Then $\delta_A^{-1}\rho^1\delta_A$ is a representation equivalent to ρ^1 which acts on V_0. Since $\delta_A^{-1}\gamma_n\delta_A = \gamma_{An}$, we can change variables to see that the projection valued measure p^1 associated with ρ^1 has multiplicity function $m^1(\chi)$ equal to the sum of the multiplicities of all the characters ψ such that $\psi(A\gamma) = \chi(\gamma)$ for all $\gamma \in \Gamma$. Since the characters ψ are points in $[-\pi,\pi)^n$ which act by $\psi(\gamma) = e^{i<\gamma,\psi>}$, this sum is over all ψ with $< \psi, A\gamma > \equiv < \chi, \gamma >$ mod 2π. Therefore, if we let σ be the representation of Γ on W_0, with multiplicity function m_σ, the decomposition $V_1 = V_0 \oplus W_0$ translates into the following information about the multiplicity function m for almost all $\chi \in [-\pi,\pi)^n$:

$$\sum_{\psi\in[-\pi,\pi)^n : A^T\psi \equiv \chi \bmod 2\pi} m(\psi) = m(\chi) + m_\sigma(\chi).$$

The statement of the lemma then follows from the observation that $m_\sigma(\chi) \geq 0$.

We would like to determine exactly which integrable functions $m : [-\pi,\pi)^n \mapsto \{0,1,2,\cdots\}$ are multiplicity functions for a GMRA. Lemma 1.3 gives the first condition m must satisfy. To describe the other condition, we will need some notation. For any set $G \subset [-\pi,\pi)^n$, we let $\tilde{G} = \{x \in \mathbb{R}^n : x = y + 2\pi\gamma \text{ for some } \gamma \in \Gamma \text{ and } y \in G\}$. Recall that $S_j = \{x \in [-\pi,\pi)^n : m(x) \geq j\}$. Following work of

Rzeszotnik and Speegle ([RS]), we let $\Delta = \cap_{p=0}^{\infty}(A^T)^p\tilde{S}_1$. Finally, for each $x \in \mathbb{R}^n$, we let $\Delta_x = \{y \in \Delta : y \equiv x \bmod 2\pi\}$. The second condition on m is then

$$\sum_{\gamma\in\Gamma} \chi_{\Delta}(x + 2\pi\gamma) \geq m(x). \tag{2}$$

We will show that taken together, conditions (1) and (2) give necessary and sufficient conditions on m for the existence of an associated GMRA. We will prove sufficiency by constructing a set $E \subset L^2(\mathbb{R}^n)$ such that $\{V_j\}$ with $\widehat{V_j} = L^2((A^T)^jE)$ forms a GMRA. We will need the following lemma, which gives the conditions such an E must satisfy.

LEMMA 1.4. *Let E be a measurable subset of $\mathbb{R}^n$ such that $E \subset A^TE$ and $\sum_{\gamma\in\Gamma}\chi_E(x + 2\pi\gamma)$ is integrable. Then $\{V_j\}$ defined by $\widehat{V_j} = L^2((A^T)^jE)$ is a (subspace) GMRA with multiplicity function $m(x) = \sum_{\gamma\in\Gamma}\chi_E(x + 2\pi\gamma)$. It is a full-space GMRA if and only if in addition,*

$$\cup_{p\in\mathbb{Z}}(A^T)^pE = \mathbb{R}^n$$

up to a set of measure zero.

Proof. Properties (1), (2) and (4) in the definition of a GMRA are immediate. Also, the representation ρ of Γ on V_0 is equivalent via the Fourier transform to multiplication by $e^{i<\cdot,\gamma>}$on $L^2(E)$. Thus, for $x \in [-\pi,\pi)^n$, $m(x) = \sum_{\gamma\in\Gamma}\chi_E(x + 2\pi\gamma)$.

To see that $\cap_{j\in\mathbb{Z}}V_j = 0$, it will suffice to show that $G \equiv \cap_{j\in\mathbb{Z}}(A^T)^jE$ has measure 0. Since $\det(A) \geq 2$ and G is invariant under A^T, we have $2\mu(G) \leq \mu(A^TG) = \mu(G)$, so that G must have measure 0 or ∞. Since m is integrable, $\mu(G) \leq \mu(E) = \int m < \infty$.

It remains to show that a collection $\{V_j\}$, that satisfies all the GMRA conditions other than the closure of the union is dense, is a full-space GMRA if and only if $\cup_{p\in\mathbb{Z}}(A^T)^p(E) = \mathbb{R}^N$. Thus, let H be the closure of the union of the V_j's, and note that H is invariant under dilation by A and translation by every element of $\mathbb{R}^N$. (It is invariant under every dyadic translation.) Write τ for the representation of $\mathbb{R}^N$ determined by translation on H. Note that $H = L^2(\mathbb{R}^N)$ if and only if τ is equivalent to the regular representation of $\mathbb{R}^N$.

Suppose first that $\{V_j\}$ is not a full-space GMRA, i.e., that τ is not equivalent to the regular representation. This implies that there exists a set $F \subseteq \mathbb{R}^N$ having positive measure such that $\widehat{f}(x) = 0$ for every $x \in F$ and $f \in H$. Since H is invariant under dilation by A, it follows in fact that F may be chosen to be invariant under dilation by A^T. We then have that $\chi_E((A^T)^p(x)) = 0$ for every $x \in F$, implying that $(A^T)^p(x) \notin E$ for every $x \in F$ and every integer p. Hence, F is disjoint from $\cup(A^T)^p(E)$.

Conversely, Suppose $F = \mathbb{R}^n \setminus \cup(A^T)^p(E)$ has positive measure. Then, any function g, whose Fourier transform is supported on F, will be orthogonal to H, implying that τ is a proper subrepresentation of the regular representation.

THEOREM 1.5. *Let $m : [-\pi,\pi)^n \mapsto \{0, 1, 2, \cdots\}$ be an integrable function. Then m is the multiplicity function for a (subspace) GMRA in $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{y\in[-\pi,\pi)^n:A^Ty\equiv x \bmod 2\pi} m(y) \geq m(x) \tag{1}$$

and

$$\sum_{\gamma\in\Gamma}\chi_\Delta(x+2\pi\gamma)\geq m(x) \tag{2}$$

for almost all $x\in[-\pi,\pi)^n$. It is the multiplicity function for a full-space GMRA if in addition

$$\cup_{p\in\mathbb{Z}}(A^T)^p\Delta=\mathbb{R}^n$$

up to a set of measure zero.

Proof. First we suppose m is associated with a GMRA. Lemma 1.3 then proves that equation (1) holds. To prove (2), fix a j and an x with $m(x)=j$. Then $x\in\tilde{S}_j\subset\tilde{S}_i$ for $i\leq j$. By Lemma 1.2, for each $i\leq j$, $\exists\gamma_i$ such that $\hat{\phi}_i(x+2\pi\gamma_i)\neq 0$. Again by Lemma 1.2, these translates $x+2\pi\gamma_1,x+2\pi\gamma_2,\cdots,x+2\pi\gamma_j$ can be taken to be distinct.

We know that we have $\widehat{V}_0\subset\widehat{V}_p$ for $p\geq 0$, and also that $\widehat{V}_p$ is spanned by functions of the form $e^{i<\cdot,(A^T)^{-p}\gamma>}\hat{\phi}_i((A^T)^{-p}x)$. Thus, since $\hat{\phi}_i\in\widehat{V}_0$, we have for any $y\in[-\pi,\pi)^n$, $\hat{\phi}_k((A^T)^{-p}y)=0\quad\forall k\implies\hat{\phi}_i(y)=0$. Applying this to $y=x+2\pi\gamma_i$ we get for each $1\leq i\leq j$, an $i'\geq 1$ such that $\hat{\phi}_{i'}((A^T)^{-p}(x+2\pi\gamma_i))\neq 0$. This in turn implies that $x+2\pi\gamma_i\in(A^T)^p\tilde{S}_{i'}\subset(A^T)^p\tilde{S}_1$ for each $p\geq 0$ and each $1\leq i\leq j$. We have thus found j distinct translates of x in Δ.

Now to prove the converse direction of the theorem, suppose m satisfies equations (1) and (2). Let $S_i=\{x\in[-\pi,\pi)^n : m(x)\geq i\}$ as before. By equation (2) we know that each $x\in S_i$ has i distinct translates $x+2\pi\gamma_1,x+2\pi\gamma_2,\cdots,x+2\pi\gamma_i$ in Δ. We will prove that there is a GMRA associated with m by building a set $E=\cup_{j=1}^\infty E_j\subset\mathbb{R}^n$ such that $E\subset A^TE$, E_j is $2\pi-$translation congruent to S_j, and the E_j are pairwise disjoint. Our GMRA will then be defined by $\widehat{V}_p=L^2((A^T)^pE)$ for $p\in\mathbb{Z}$.

First we build a set E_1 such that $E_1\subset A^TE_1$ and E_1 is $2\pi-$translation congruent to S_1. Let F be an open neighborhood of the origin in $\mathbb{R}^n$ such that $F\subseteq[-\pi,\pi)^n$ and $F\subseteq A^TF$. Such a neighborhood can be found by putting A^T into canonical form as a complex matrix, applying the resulting change of basis matrix to a neighborhood of the origin in $\mathbb{C}^n$, and then taking real parts. This set F has the property that $\cup_{p\in\mathbb{Z}}(A^T)^pF=\mathbb{R}^n$. (see e.g. [GH]). Let $E_{1,0}=\Delta\cap F$. We use $E_{1,0}$ to recursively define a partition of S_1 as follows: Let $S_{1,0}=E_{1,0}$, and for $j\geq 1$, define $S_{1,j}=\{x\in S_1\setminus\cup_{k=1}^{j-1}S_{1,k} : \Delta_x\cap(A^T)^jE_{1,0}\neq\emptyset\}$. By equation (2), $\cup_{j=0}^\infty S_{1,j}=S_1$.

We will use the partition $\{S_{1,j}\}$ to recursively construct disjoint pieces $E_{1,j}$ of E_1, with $E_{1,j}$ $2\pi-$translation congruent to $S_{1,j}$. We already have $E_{1,0}=S_{1,0}$. Now suppose we have constructed $E_{1,k}$ for $k<j$. If $S_{1,j}$ is empty, take $E_{1,j}=\emptyset$. If $x\in S_{1,j}$, then there is a translate of x in $\Delta\cap(A^T)^jE_{1,0}\subset A^T(\Delta\cap(A^T)^{j-1}E_{1,0})$, so $x=A^Tz$ for some $z\in\Delta\cap(A^T)^{j-1}E_{1,0}$. If we let $y\in[-\pi,\pi)^n$ be congruent to z mod 2π, then $y\in S_{1,j-1}$, since if $y\in S_{1,k}$ for $k<j-1$, we would have $x\in S_{1,k+1}$, contradicting $x\in S_{1,j}$. If there is more than one z with the properties we have described, let y_x be the point of smallest absolute value in $S_{1,j-1}$ with $A^Ty_x\equiv x$ mod 2π. Let $y_x+2\pi\gamma_{1,y_x}$be the translate of y_x in $E_{1,j-1}$. Choose $\gamma_{1,x}\in\Gamma$ so that $x+2\pi\gamma_{1,x}=A^T(y_x+2\pi\gamma_{1,y_x})$. Take $E_{1,j}=\{(x+2\pi\gamma_{x,1}) : x\in S_{1,j}\}$.

We finally let $E_1=\cup_{j=0}^\infty E_{1,j}$. Since $E_{1,j}$ is $2\pi-$translation congruent to $S_{1,j}$ we immediately see that E_1 is 2π-translation congruent to S_1. Since $E_{1,j}\subset A^TE_{1,j-1}$

for $j > 0$, and $E_{1,0} \subset A^T E_{1,0}$ we have that $E_1 \subset A^T E_1$. This completes the construction of E_1.

Now suppose we have constructed pairwise disjoint sets $E_1, E_2, \cdots E_{N-1}$ such that E_j is 2π-translation congruent to S_j, and $\cup_{j=1}^{N-1} E_j \subset A^T(\cup_{j=1}^{N-1} E_j)$. If $S_N = \emptyset$, then $S_j = \emptyset$ for all $j \geq N$, and we take $E_j = \emptyset$ for $j \geq N$ as well. If not, we proceed to construct E_N from $\cup_{j=1}^{N-1} E_j$ by a technique that is similar to the method used to build E_1 from $E_{1,0}$. Thus we take $E_{N,0} = \cup_{k=1}^{N-1} E_k$. Note that for $N > 1$, $E_{N,0}$ will be used to construct E_N, but will not be contained in E_N. As before, we first recursively define a partition of S_N. Since $E_{N,0}$ is to be disjoint from E_N, we let $S_{N,0} = \emptyset$. For $j \geq 1$, define $S_{N,j} = \{x \in S_N \backslash (\cup_{k=1}^{j-1} S_{N,k}) : card(\Delta_x \cap (A^T)^j E_{N,0})) \geq N\}$. By equation (2), $\cup_{j=1}^{\infty} S_{N,j} = S_N$.

We will use the partition $\{S_{N,j}\}$ to recursively construct disjoint pieces $E_{N,j}$ of E_N, with $E_{N,j}$ $2\pi-$translation congruent to $S_{N,j}$ for $j > 0$. First we construct $E_{N,1}$. If $x \in S_{N,1}$ then there are at least N translates of x in $\Delta \cap (A^T) E_{N,0}$. However, since in each E_j there is at most one point congruent to x, there are at most $N-1$ translates of x in $E_{N,0} = \cup_{k=1}^{N-1} E_k$. Choose $\gamma_{x,N} \in \Gamma$ so that $x + 2\pi\gamma_{x,N}$ is the smallest in absolute value among all the translates of x that are in $\Delta \cap (A^T) E_{N,0}$ but are not in $E_{N,0}$. Let $E_{N,1} = \{x + 2\pi\gamma_{x,N} : x \in S_{N,1}\}$, and note that $E_{N,1}$ is disjoint from $E_{N,0}$.

Now take $j \geq 2$, and suppose we have constructed pairwise disjoint sets $E_{N,k}$ for $k < j$. If $x \in S_{N,j}$, then there are at least N translates of x, call them $x_1, x_2, \cdots$ in $\Delta \cap (A^T)^j E_{N,0} \subset A^T(\Delta \cap (A^T)^{j-1} E_{N,0})$. We claim that $(A^T)^{-1} x_1, (A^T)^{-1} x_2, \cdots$ are all in the same congruence class y mod 2π. If not, when we apply equation (1) to x we get at least two nonzero terms on the left-hand side. If we let $q_1, q_2, \cdots q_k$ be the distinct points in $[-\pi, \pi)^n$ which correspond to these nonzero terms, equation (1) gives $\sum_{l=1}^{k} m(q_l) \geq N$. Since each E_j contains at most one translate of any given point, each q_l has $\min(m(q_l), N-1)$ translates in $E_{N,0} = \cup_{k=1}^{N-1} E_k$. So, since $\sum_{l=1}^{k} \min(m(q_l), N-1) \geq N$, there are at least N translates of the q_j's in $E_{N,0}$. But, since $x \notin S_{N,1}$ we know that $\Delta \cap (A^T) E_{N,0}$ cannot contain N different translates of x.

If y_x is the single congruence class of $(A^T)^{-1} x_1, (A^T)^{-1} x_2, \cdots$ then we have shown that the cardinality of $\Delta_{y_x} \cap (A^T)^{j-1} E_{N,0}$ is at least N, so $y_x \in S_{N,j-1}$. Let $y_x + 2\pi\gamma_{y_x,N}$be the translate of y_x in $E_{N,j-1}$. Choose $\gamma_{x,N} \in \Gamma$ so that $A^T(y_x + 2\pi\gamma_{y_x,N}) = x + 2\pi\gamma_{x,N}$. Since $y_x + 2\pi\gamma_{y_x,N} \notin E_{N,0}$ and $E_{N,0} \subset A^T(E_{N,0})$ we have $x + 2\pi\gamma_{x,N} \notin E_{N,0}$. Define $E_{N,j} = \{(x + 2\pi\gamma_{x,N}) : x \in S_{N,j}\}$, and $E_N = \cup_{j=1}^{\infty} E_{N,j}$. As in the case of E_1, since $E_{N,j}$ is $2\pi-$translation congruent to $S_{N,j}$, we immediately see that E_N is 2π-translation congruent to S_N. Also, since $E_{N,j} \subset A^T E_{N,j-1}$ for $j > 0$, and $E_{N,0} = \cup_{j=1}^{N-1} E_j$. we have that $(\cup_{j=1}^{N} E_j) \subset A^T(\cup_{j=1}^{N} E_j)$.

We let $E = \cup_{j=1}^{\infty} E_j$. Our inductive procedure guarantees that $E \subset A^T E$, that E_j is 2π-translation congruent to S_j, and that the E_j are pairwise disjoint. Let $\widehat{V_p} = L^2((A^T)^p E)$, $p \in \mathbb{Z}$. It follows from Lemma 1.4 that $\{V_p\}$ is a (subspace) GMRA. If, in addition, every $x \in \mathbb{R}^n$ is of the form $x = (A^T)^r y$ for some $y \in \Delta$ and $r \in \mathbb{Z}$, write $y = (A^T)^s w$ for $w \in F$ and $s \geq 0$ to obtain $x = (A^T)^{r+s} w$, with $w \in \Delta \cap F \subset E$. Again by Lemma 1.4, we see that $\{V_p\}$ is a full-space GMRA in this case.

Given a multiplicity function that satisfies conditions (1) and (2), there may be many associated GMRA's. For example, $m \equiv 1$ is associated to all possible MRA's.

For each allowable multiplicity function, the proof of Theorem 1.5 gives an explicit technique for building one of the associated GMRA's. By an argument similar to Lemma 1.1, we can also use the construction in the proof to build a tight frame for our GMRA, by taking $\hat{\phi}_j = \chi_{E_j}$.

GMRA EXAMPLES. For the trivial case of $m \equiv 1$, conditions (1) and (2) are always satisfied, so the proof of Theorem 1.5 gives a method for building a full-space MRA for an arbitrary dilation in any dimension. If $[-\pi,\pi)^n \subset A^T([-\pi,\pi)^n)$, then the construction ends with $E_{1,0}$, and we have that $\widehat{V}_j = L^2((A^T)^j[-\pi,\pi)^n)$ and that V_0 has scaling function ϕ with $\hat{\phi} = \chi_{[-\pi,\pi)^n}$. If, on the other hand, $[-\pi,\pi)^n \not\subset A^T([-\pi,\pi)^n)$, then the construction must take $E_{1,0} = F$ to be a proper subset of $[-\pi,\pi)^n$, and the other $E_{1,k}$'s are not all empty. For example, if in $L^2(\mathbb{R}^2)$, $A = \begin{pmatrix} 2 & 3 \\ -2 & -2 \end{pmatrix}$ then F cannot be taken to be $[-\pi,\pi)^2$, but it can be taken to be the parallelogram with vertices at $\pm(2,1)$, $\pm(2,3)$. The construction in the proof of Theorem 1.5 then leads to a polygon E_1 with vertices at $\pm(\pi,\pi-4)$, $\pm(\pi,3\pi-4)$, $\pm(4-\pi,\pi)$, and $\pm(\pi-4,\pi)$. We have an MRA with $\widehat{V}_0 = L^2(E_1)$ and scaling function ϕ with $\hat{\phi} = \chi_{E_1}$.

If we pick a proper subset $B \subset [-\pi,\pi)^n$, then $m = \chi_B$ may or may not satisfy the two conditions of Theorem 1.5. However, it will if we take B to be a neighborhood of the origin such that $B \subseteq A^T B$. The theorem then gives a full-space GMRA with $\widehat{V}_0 = L^2(B)$, and a frame (no scaling function exists) consisting of translates of ϕ with $\hat{\phi} = \chi_B$.

To get GMRA's with higher multiplicities, we can, for example take $m(x) = \left\{\begin{smallmatrix} 2 \text{ if } x\in B \\ 1 \text{ if } x\notin B \end{smallmatrix}\right.$, where B is any subset of $[-\pi,\pi)^n$. Since $\Delta = \mathbb{R}^n$, condition (2) is easily satisfied. Condition (1) follows since there are always at least two terms on the left hand side, each of which is at least 1. For example, if we again let $A = \begin{pmatrix} 2 & 3 \\ -2 & -2 \end{pmatrix}$ in $L^2(\mathbb{R}^2)$, and let B be the parallelogram with vertices at $\pm(2,1)$, $\pm(2,3)$, then the construction gives E_1 as before a polygon with vertices at $\pm(\pi,\pi-4)$, $\pm(\pi,3\pi-4)$, $\pm(4-\pi,\pi)$, and $\pm(\pi-4,\pi)$. The next level of construction, E_2, is the union of two parallelograms R and $-R$, where R has vertices $(2\pi-2, 2\pi-1)$, $(2\pi-2, 2\pi-3)$, $(2\pi, 2\pi-1)$ and $(2\pi, 2\pi+1)$. The GMRA, given by $\widehat{V}_1 = L^2(E_1 \cup E_2)$, has frame consisting of translates of ϕ_1 and ϕ_2 with $\hat{\phi}_1 = \chi_{E_1}$ and $\hat{\phi}_2 = \chi_{E_2}$.

Wavelets and the multiplicity function

We now apply the results of the previous section to wavelets. In our context, a wavelet is defined as follows:

DEFINITION. A *wavelet* is a vector $\psi \in L^2(\mathbb{R}^n)$ such that the collection $\{\delta^j(\gamma(\psi))\}$, for $j \in \mathbb{Z}$ and $\gamma \in \Gamma$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. The vector ψ is called a *subspace wavelet* if these vectors form an orthonormal basis for a subspace of $L^2(\mathbb{R}^n)$.

It is not difficult to show that every wavelet has an associated GMRA, with V_j equal to the closed linear span of the vectors $\delta^k(\gamma(\psi))$ for $k < j$ and $\gamma \in \Gamma$. If we write $V_1 = V_0 \oplus W_0$, translates of the wavelet ψ then give an orthonormal basis for W_0. The representation of Γ on W_0 has multiplicity function identically equal to 1.

Thus if we analyze the representation of Γ on each piece as in Lemma 1.3, we get the *consistency equation*:

$$(1') \qquad \sum_{y\in[-\pi,\pi)^n:A^Ty\equiv x \bmod 2\pi} m(y) = m(x) + 1$$

This *consistency equation* can be shown to give a necessary and sufficient condition for a GMRA to have an associated wavelet:

THEOREM 2.1. *If ψ is a wavelet in $L^2(\mathbb{R}^n)$, then the collection of subspaces $\left\{V_j = \overline{sp\{\delta^k(\gamma(\psi)) : k < j, \gamma \in \Gamma\}}\right\}$ is a generalized multiresolution analysis, whose multiplicity function m satisfies the following consistency equation a.e. on $[-\pi,\pi)^n$:*

$$(1') \qquad \sum_{y\in[-\pi,\pi)^n:A^Ty\equiv x \bmod 2\pi} m(y) = m(x) + 1$$

Conversely, if $\{V_j\}$ is a generalized multiresolution analysis of $L^2(\mathbb{R}^n)$ whose multiplicity function m is finite a.e. and satisfies the consistency equation $(1')$ a.e. on $[-\pi,\pi)^n$, then there exists a vector ψ in the subspace W_0 that forms a wavelet for $L^2(\mathbb{R}^n)$, with $\left\{V_j = \overline{sp\{\delta^k(\gamma(\psi)) : k < j, \gamma \in \Gamma\}}\right\}$. The analogous statements hold for subspace GMRA's and subspace wavelets.

Proof. see [BMM]

Putting this result together with Theorem 1.5, we have the following necessary and sufficient conditions for an integrable function $m : [-\pi,\pi)^n \mapsto \{0,1,2,\cdots\}$ to determine a wavelet:

THEOREM 2.2. *Let m be an integrable function mapping $[-\pi,\pi)^n$ into $\{0,1,2,\cdots\}$. There exists a (subspace) wavelet whose multiplicity function is m if and only if*

$$(1') \qquad \sum_{y\in[-\pi,\pi)^n:A^Ty\equiv \chi \bmod 2\pi} m(y) = m(x) + 1$$

and

$$(2) \qquad \sum_{j\in\Gamma} \chi_\Delta(x + 2\pi\gamma) \geq m(x).$$

Proof. The statement follows immediately from Theorem 1.5 and Theorem 2.1.

Given a function m which satisfies these two properties, we can build a wavelet by first using the proof of Theorem 1.5 to build a set E such that $\{V_j\}$ given by $\widehat{V_j} = L^2((A^T)^jE)$ is an associated GMRA. Then since $\widehat{W_0} = L^2(W)$ where $W = A^TE \setminus E$, we can take $\hat{\psi} = \chi_W$. A wavelet ψ such that $|\hat{\psi}| = \chi_W$ for some set $W \subset \mathbb{R}^n$ is called a *minimally supported frequency* (MSF) wavelet; W is called a *wavelet set.* Thus we have the following corollary:

COROLLARY 2.3. *If m is an integrable function which is the multiplicity function for any wavelet in $L^2(\mathbb{R}^n)$ then it is the multiplicity function for an MSF wavelet in $L^2(\mathbb{R}^n)$.*

REMARK. *For dilation by 2 wavelets in $L^2(\mathbb{R}^1)$, the multiplicity function has been shown (* [W]*) to equal the dimension function*

$$D_\psi(x) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(x + 2\pi k))|^2$$

which was introduced by Auscher in [A]*. Calogero* [Ca] *has generalized the dimension function to dilation by arbitrary expansive matrices in $L^2(\mathbb{R}^n)$, and used it to characterize MRA wavelets in this more general context. Recently, Rzeszotnik and Speegle (* [RS]*) gave necessary and sufficient conditions for a function mapping $[-\pi, \pi)$ to $\{0, 1, 2, \cdots\}$ to be the dimension function of a dilation by 2 wavelet in $L^2(\mathbb{R}^1)$. Their conditions are equivalent to those of Theorem 2.2 for full space wavelets in this special case, and they obtain the corresponding special case of Corollary 2.3.*

WAVELET EXAMPLES. The MRA examples explored in the previous section satisfies the consistency equation $(1')$ if and only if $\det(A) = 2$, since in this case there are exactly two terms on the left-hand side. Thus, in this case, our construction in the previous section can also be used as construction of MSF wavelets. In particular, this gives a new proof of a result by Gu and Han [GH] that MRA MSF wavelets always exist for expansive matrices of determinant 2. The particular higher multiplicity example of a GMRA given in Section 1, $m(x) = \begin{cases} 2 \text{ if } x \in B \\ 1 \text{ if } x \notin B \end{cases}$, for B a measurable subset of $[-\pi, \pi)^n$, can never have an associated wavelet. This can be seen by integrating the consistency equation to get $\mu(E) = \frac{(2\pi)^n}{\det(A)-1} \leq (2\pi)^n$. Thus a wavelet multiplicity function that takes on values greater than 1 must also take on the value 0.

New examples of wavelet sets with varying multiplicities can be easily found by building a solution m to the consistency equation and then checking that it satisfies equation (2) as well. In [BMM] we use the consistency equation in this way to to give a technique for building all wavelet sets in $L^2(\mathbb{R}^n)$, and go on to construct several examples.

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