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Simple Wavelet Sets for Scalar Dilations in $\mathbb{R}^2$

Kathy D. Merrill

Department of Mathematics, Colorado College, Colorado Springs, CO 80903, USA
kmerrill@coloradocollege.edu

Summary. Wavelet sets for dilation by any scalar $d > 1$ in $L^2(\mathbb{R}^2)$ are constructed that are finite unions of convex polygons. Such simple wavelet sets for dilation by 2 were widely conjectured to be impossible. The examples are built using the generalized scaling set technique of Baggett *et.al.* [3]. Generalizations to other expansive dilations in $L^2(\mathbb{R}^2)$ are discussed.

Dedicated to Larry Baggett for his contributions of multiplicity and consistency.

1.1 Introduction

The term "wavelet set" was coined by Dai and Larson [13] in the late 1990's to describe a set W such that χ_W, the characteristic function of W, is the Fourier transform of an orthonormal wavelet on $L^2(\mathbb{R}^n)$. Here, by orthonormal wavelet they meant a single function ψ whose successive dilates by a scalar of all translates by the integer lattice, form an orthonormal basis for $L^2(\mathbb{R})$. This definition was later generalized to higher dimensions and to allow for other dilation and translation sets. In this paper, we will restrict our attention to translations by the integer lattice in $\mathbb{R}^n$, and dilations by an expansive real-valued (sometimes restricted to be integer-valued) $n \times n$ matrix A, where by expansive we mean that all the eigenvalues have absolute value greater than 1. At about the same time as the Dai/Larson paper, Fang and Wang [18] first used the term MSF wavelet (minimally supported frequency wavelet) to describe wavelets whose Fourier transforms are supported on sets of the smallest possible measure, and noted that such a wavelet ψ would necessarily have $|\hat{\psi}|$ a characteristic function. Since it is also true that any multiple of an MSF wavelet by a function of absolute value 1 is still an MSF wavelet, MSF wavelets can be characterized as precisely those wavelets associated with wavelet sets.

Even before this formal beginning of their study, wavelet sets and MSF wavelets were important to wavelet theory as a source of examples. One of the simplest and earliest wavelets to be studied was the Shannon or Littlewood-Paley wavelet for dilation by 2 in $L^2(\mathbb{R})$, an MSF wavelet with wavelet set $W = [-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1)$ (see [16]). This wavelet derives its names from its important connections to the Shannon Sampling Theorem [32] and to Littlewood-Paley theory [28]. It can be thought of as a complementary example to another early and simple example for dilation by 2 in $L^2(\mathbb{R})$: the Haar wavelet [21], $\psi = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}$. While the Haar wavelet is well-localized but not smooth, the Shannon is smooth but not well-localized. These two examples each characterize one of the two properties desired of wavelets for the purpose of applications. In the mid 1980's, Meyer [29], Battle [5] and Lemarie [26] found wavelets for dilation by 2 in $L^2(\mathbb{R})$ that have both good smoothness and good localization properties, culminating in Daubechies' [17] construction of a single wavelet that was both smooth and compactly supported. In contrast, MSF wavelets in general are not directly useful for applications, in that their discontinuous Fourier transforms mean that they cannot be well-localized.

However, the importance of MSF wavelets as a source of examples and counterexamples has continued throughout wavelet history. A famous example due to Journé [16] first showed that not all wavelets have an associated structure called a multiresolution analysis (MRA). This structure, which grades $L^2(\mathbb{R}^n)$ into a nested sequence of closed subspaces controlled by level of dilation, requires that the base space has an orthonormal basis consisting of translates of a single function called a scaling function. The Journé wavelet fails this last requirement. The fact that this counterexample is an MSF wavelet is a symptom of a larger fact: Auscher proved in [1] that every wavelet whose Fourier transform satisfies a weak smoothness and decay condition must be associated with an MRA. The discovery of a non-MRA wavelet gave an important push to the development of more general structures such as frame multiresolution analyses (FMRA's) (see e.g. [8],[31]) and generalized multiresolution analyses (GMRA's) [3].

Counterbalancing the desire for smooth, well-localized wavelets and MRA wavelets is a desire for single wavelets. If a set of k functions $\{\psi_1, \cdots, \psi_k\}$ has the property that their dilates of translates form an orthonormal basis for $L^2(\mathbb{R}^n)$, we say that $\{\psi_1, \cdots, \psi_k\}$ is a k-wavelet. For example, the simplest wavelets to build for dilation by 2 in $L^2(\mathbb{R}^n)$ are $(2^n - 1)$-wavelets, formed using tensor products of wavelets and scaling functions from one-dimensional space. Moreover, it was shown in 1995 (see [1], [2], or [19]), that an MRA wavelet for dilation by an expansive integer valued matrix A in $L^2(\mathbb{R}^n)$ must be a $(|\det A|-1)$-wavelet. For applications set in high-dimensional spaces, though, it is desirable to keep the number of wavelets under control, and information about how many wavelets are required is of theoretical interest. Until the late 1990s, it was an open question whether single wavelets (necessarily non-MRA except for the determinant 2 case) existed in dimension higher than 1. This question was settled by Dai, Larson and Speegle [14], who showed that wavelet sets, and thus single wavelets, exist for an arbitrary expansive matrix in any dimension. Here again, it was MSF wavelets that provided an essential example.

In addition to their usefulness as examples or counterexamples, MSF wavelets have been used as building blocks for more well-localized wavelets. Hernandez, Wang and Weiss ([22], [23]) smoothed the filters associated with MSF wavelets for dilation by 2 in $L^2(\mathbb{R})$ to produce wavelets whose Fourier transforms were arbitrarily smooth. In the same spirit, Bownik and Speegle [11] used smoothing of MSF wavelet filters to show that $(|\det A| - 1)$-wavelets with compactly supported smooth Fourier transforms exist for every expansive integral matrix in $L^2(\mathbb{R}^2)$. By the theorem of Auscher mentioned above, the Fourier transforms of non-MRA wavelets cannot be smoothed while retaining their non-MRA and orthonormal characteristics. However, the non-MRA wavelet based on the Journé wavelet set has been smoothed to give a non-MRA Parseval wavelet (dilates of translates form a normalized tight frame rather than an orthonormal basis) whose Fourier transform is both smooth and rapidly decaying [4]. Using a different approach than that of smoothing filters, Dai and Larson [13] developed a procedure they call interpolation to find well-behaved wavelets that lie between two MSF wavelets in an operator theoretic sense. Finally, Lim, Packer and Taylor [27] used wavelet sets as building blocks in a different sense, as a space over which to decompose wavelet representations as direct integrals of irreducibles.

Thus, for wavelet theory, wavelet sets are a desirable commodity. Examples are relatively easy to construct in one dimension, and many appear in the literature, mostly for dilation by 2, but also (see e.g. [3], [13]) for arbitrary dilations. Higher dimensional examples are more difficult to find. The first two dimensional examples for dilation by 2 appeared in the late 1990's in [15], [33] and [36]. Shortly after these examples appeared, general construction techniques were introduced by Baggett, Medina and Merrill [3] and Benedetto and Leon [6] that can be used for arbitrary expansive matrix dilations in any dimension. All the construction techniques are iterative, and until now, all the wavelet set examples for scalar dilations in dimension 2 and higher showed the fingerprint of these iterative procedures, and thus have had a fractal-like structure. (See e.g. Figure 1.1 below for previous examples of dilation by 2 wavelet sets in $L^2(\mathbb{R}^2)$.) In fact, the only previously known simple wavelet sets in dimension greater than one are those for dilation by matrices of determinant ± 2 in dimension 2. (See Section 1.4.) Many researchers believed that dimension 2 or higher dyadic wavelet sets, in particular, must necessarily have a complicated geometric structure. For example, in [9], Benedetto and Sumetkijakan showed that a wavelet set for dilation by 2 in $\mathbb{R}^n$, $n \geq 2$, cannot be the union of n or fewer convex sets, and conjectured that it could not be the union of a finite number of convex sets. Soardi and Wieland in [33] made the weaker conjecture that a dilation by 2 wavelet set in dimension greater than 1 could not be the finite union of polygons.

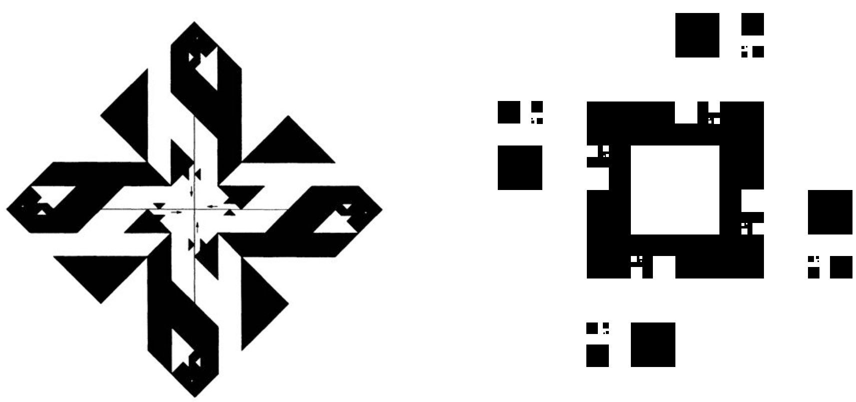


Fig. 1.1. Hole in the middle wavelet set of [33] and Windmill wavelet set of [3] and[6]

The primary purpose of this paper is to construct counterexamples to these conjectures for dilation by any real scalar $d > 1$ in $L^2(\mathbb{R}^2)$. In Section 1.2, we develop the construction technique (based on [3]) that we used to build our examples. In Section 1.3, we first build one-dimensional examples needed for the two-dimensional constructions, and then build the two-dimensional examples themselves. Finally, Section 1.4 contains some possible directions for generalizations.

1.2 The Construction Technique

A well known characterization, expressed in the following theorem, describes wavelet sets as precisely those sets that tile $\mathbb{R}^n$ both by translation and dilation.

Theorem 1. *A measurable set $W \subset \mathbb{R}^n$ is a wavelet set for dilation by an invertible real-valued matrix A if and only if*

$$\sum_{k \in \mathbb{Z}^n} \chi_W(x+k) = 1 \qquad a.e.\ x \in \mathbb{R}^n \tag{1.1}$$

$$\sum_{j \in \mathbb{Z}} \chi_W(A^{*j}x) = 1 \qquad a.e.\ x \in \mathbb{R}^n. \tag{1.2}$$

Proof. See, e.g. [13].

This theorem gives an easy method to verify a set's claim to be a wavelet set, but no easy method to discover them. In this section we outline the construction technique that was used to discover the wavelet sets described in this paper. This method was first developed in [3] for integer-valued matrix dilations, as a technique for building all wavelet sets. The original technique uses the theory of a *generalized multiresolution analysis (GMRA)*, a collection of nested subspaces $\{V_j\}$ that can be built from any wavelet ψ by letting $V_j = \overline{span}\{\psi(A^k \cdot -l) : k < j, l \in \mathbb{Z}^n\}$. However, as noted in [10], the essential ideas behind the wavelet set construction of [3] can be described independently of GMRA theory, using only the characterization of wavelet sets given by Theorem 1. In order to make the presentation given here self-contained, we will take the latter approach, and point out the connection to GMRA's only briefly in passing. This approach has the additional advantage of allowing the dilation matrix A to be non-integer valued. Although we will be mostly interested in the case where $n = 2$ and the dilation matrix is $\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$, we present the technique for a general expansive matrix in any dimension, as it introduces no additional complications.

The construction depends upon first building a *generalized scaling set*. Motivated by the idea of a GMRA, or in applications by the idea of controlling the level of detail considered for an image described by a wavelet, it is useful to look at the subspaces obtained from all translates of ψ, but only those dilates less than a fixed cutoff. The idea of a generalized scaling set comes from doing this for a wavelet ψ with $\hat{\psi} = \chi_W$. Note that in this case, the Fourier transforms of just the translates $\{\psi(\cdot - l) : l \in \mathbb{Z}^n\}$, form an orthonormal basis for $L^2(W)$, and thus the Fourier transforms of all the negative dilates of translates, $\{\psi(A^k \cdot -l) : l \in \mathbb{Z}^n,\ k < 0\}$ form an orthonormal basis for $L^2(E)$, where $E = \cup_{j<0} A^{*j} W$.

Definition 1. *A set $E \subset \mathbb{R}^n$ is called a* generalized scaling set *for dilation by A if $E = \cup_{j<0} A^{*j} W$ for some wavelet set W, or equivalently, if $E \subset A^*E$ and $A^*E \setminus E$ is a wavelet set.*

The equivalence of these two formulations is easily established (see e.g.[10]), and means that the construction of a scaling set will lead immediately to a wavelet set.

If $E \subset \mathbb{R}^n$ is a generalized scaling set for dilation by A, we can use the fact that W tiles $\mathbb{R}^n$ by both translations and dilations to see that E must satisfy the following *consistency equation*, which is a special case of the consistency equation satisfied by the multiplicity function of GMRA theory [3].

$$\begin{aligned}
\sum_{j\in\mathbb{Z}^n} \chi_E\left(A^{*-1}(x+j)\right) &= \sum_{j\in\mathbb{Z}^n} \chi_{A^*E}(x+j) \\
&= \sum_{j\in\mathbb{Z}^n} \chi_{\cup_{k\le 0} A^{*k}W}(x+j) \\
&= \sum_{j\in\mathbb{Z}^n} \sum_{k\le 0} \chi_{A^{*k}W}(x+j) \\
&= \sum_{k\le 0} \sum_{j\in\mathbb{Z}^n} \chi_{A^{*k}W}(x+j) \\
&= \sum_{j\in\mathbb{Z}^n} \chi_W(x+j) + \sum_{k<0} \sum_{j\in\mathbb{Z}^n} \chi_{A^{*k}W}(x+j) \\
&= 1 + \sum_{j\in\mathbb{Z}^n} \chi_{\cup_{k<0} A^{*k}W}(x+j) \\
&= 1 + \sum_{j\in\mathbb{Z}^n} \chi_E(x+j) \text{ a.e.}.
\end{aligned}$$

The following theorem uses the consistency equation to give sufficient conditions for a set to be a generalized scaling set.

Theorem 2. *Suppose that A is an invertible real $n\times n$ matrix and that the measurable set $E\subset\mathbb{R}^n$ is invariant under A^{*-1}; contains a neighborhood of the origin; and that χ_E satisfies the consistency equation*

$$1 + \sum_{j\in\mathbb{Z}^n} \chi_E(x+j) = \sum_{j\in\mathbb{Z}^n} \chi_E\left(A^{*-1}(x+j)\right) \text{ a.e.}. \tag{1.3}$$

Then $W = A^*E\setminus E$ is a wavelet set for dilation by A.

Proof. The consistency equation 1.3 can be rewritten:

$$\begin{aligned}
1 &= \sum_{j\in\mathbb{Z}^n} \chi_E\left(A^{*-1}(x+j)\right) - \sum_{j\in\mathbb{Z}^n} \chi_E(x+j) \\
&= \sum_{j\in\mathbb{Z}^n} \chi_{A^*E}(x+j) - \sum_{j\in\mathbb{Z}^n} \chi_E(x+j) \\
&= \sum_{j\in\mathbb{Z}^n} \chi_{A^*E\setminus E}(x+j) \\
&= \sum_{j\in\mathbb{Z}^n} \chi_W(x+j) \text{ a.e.}.
\end{aligned}$$

This shows that W tiles $\mathbb{R}^n$ by translations.

Since A^* is one-to-one and E is invariant under A^{*-1}, we have that the dilates of W are disjoint. Because E contains a neighborhood of the origin, we can conclude that the dilates of W will (up to measure 0) cover $\mathbb{R}^n$. The result then follows by Theorem 1.

Theorem 2 is the basis of our construction technique. We use it to build a set $E = \sqcup_{i=1}^{\infty} E_i$ that is guaranteed to be a generalized scaling set as follows. We first build a set E_1 that contains a neighborhood of the identity and satisfies

$$1 = \sum_{j\in\mathbb{Z}^n} \chi_{E_1}\left(A^{*-1}(x+j)\right) \ a.e.. \tag{1.4}$$

We then recursively build disjoint sets E_i such that

$$A^{*-1}E_i \subset \cup_{k\leq i+1} E_k \tag{1.5}$$

and

$$\sum_{j\in\mathbb{Z}^n} \chi_{E_i}(x+j) = \sum_{j\in\mathbb{Z}^n} \chi_{E_{i+1}}\left(A^{*-1}(x+j)\right) \ a.e.. \tag{1.6}$$

The disjoint union of these sets, $E = \sqcup_{i=1}^{\infty} E_i$ will then satisfy all three hypotheses of Theorem 2, and thus will be a scaling set for dilation by A. This technique is described in [3] in terms of two measurable one-to-one maps T, taking $[-\frac{1}{2}, \frac{1}{2})^n$ to E_1, and T', taking E_i to E_{i+1}. Note that both maps take a point x to a translate of $A^{*-1}x$ by the lattice $A^{*-1}\mathbb{Z}^n$. The requirement (1.5) demands that $T'(x) = A^{*-1}x$ whenever such an assignment does not violate the condition that the E_i be disjoint.

1.3 The Examples

Our ultimate goal in this section is to use the technique outlined in Section 1.2 to build dilation d wavelet sets in $L^2(\mathbb{R}^2)$ that are finite unions of convex polygons. It will be useful to first carry out a similar construction in $L^2(\mathbb{R}^1)$, building dilation d wavelet sets that are finite unions of intervals. The key strategy in one-dimension is to pick an interval E_1 in such a way that some later interval E_i will be adjacent to it. This can be done in many ways; we concentrate here on constructions that most easily generalize to $L^2(\mathbb{R}^2)$. Note that Theorem 1 and the construction technique of Section 1.2 only determine wavelet sets up to a set of measure 0. Thus, endpoints of intervals in the one-dimensional constructions, or edges of polygons in the two-dimensional constructions can be dealt with arbitrarily. We take as a convention using intervals closed on the left and open on the right, and appropriate generalizations to $\mathbb{R}^2$.

First we note that any interval of the form $E_1 = [c - \frac{1}{2d}, c + \frac{1}{2d})$, with $-\frac{1}{2d} < c < \frac{1}{2d}$, will contain a neighborhood of 0 and satisfy Equation 1.4. To form E_2, we will then split E_1 in half and use a translate of its dilate by $-\frac{l}{d}$ for the left half, and a translate by $\frac{k}{d}$ for the right half, where $l, k \in \mathbb{Z}$ will be determined later. That is, we let $E_2 = \left[\frac{c}{d} - \frac{1}{2d^2} - \frac{l}{d}, \frac{c}{d} - \frac{l}{d}\right) \cup \left[\frac{c}{d} + \frac{k}{d}, \frac{c}{d} + \frac{1}{2d^2} + \frac{k}{d}\right)$, which can be seen to satisfy Equation 1.6 and also (1.5). For E_3, we will take the dilate $E_3 = \frac{1}{d}E_2$, which also satisfies Equation 1.6 and will be required by (1.5). Now we go back and choose c, l, k such that the two subintervals of E_3 are adjacent to the the two subintervals of E_1. That is, we require

$$\frac{1}{d}\left(\frac{c}{d} + \frac{k}{d}\right) = c + \frac{1}{2d} \tag{1.7}$$

and

$$\frac{1}{d}\left(\frac{c}{d} - \frac{l}{d}\right) = c - \frac{1}{2d} \tag{1.8}$$

A simultaneous solution to these two equations requires that $k + l = d$, and thus that d is an integer. In that case, one solution is to take $l = d - 1$, $k = 1$, and $c = \frac{2-d}{2d^2-2}$. With this choice, we then build the rest of the sets E_i, for $i \geq 4$ to continue this adjacency. That is, by defining T' to be the same on all odd E_i as it is on E_1, we use the adjacency of E_3 to E_1, to force E_4 to be adjacent to E_2, and so on. Similarly, we define T' to be the same on all the even E_i, namely $T'(x) = x/d$. (This last definition is actually forced by (1.5).)

The resulting scaling set for dilation by any integer $d >\geq 2$ consists of three intervals: one formed from E_1, which is centered at c and has length $\frac{d}{d^2-1}$; one formed from the right half of E_2, so with left endpoint at $\frac{c}{d} + \frac{k}{d}$ and length $\frac{1}{2d^2-2}$; and one formed from the left half of E_2, which gives the previous interval shifted to the left by 1. That is,

$$E = \left[-\frac{d}{d+1}, \frac{2d-1}{2d^2-2} - 1\right) \cup \left[-\frac{1}{d+1}, \frac{1}{d^2-1}\right) \cup \left[\frac{2d-1}{2d^2-2}, \frac{d}{d^2-1}\right), \tag{1.9}$$

so that we have a wavelet set for dilation by an arbitrary integer d given by

$$W = \left[-\frac{d^2}{d+1}, \frac{2d^2-d}{2d^2-2} - d\right) \cup \left[\frac{2d-1}{2d^2-2} - 1, -\frac{1}{d+1}\right) \cup \left[\frac{1}{d^2-1}, \frac{2d-1}{2d^2-2}\right) \cup \left[\frac{2d^2-d}{2d^2-2}, \frac{d^2}{d^2-1}\right).$$

We can alter this construction slightly as follows. Using the same values for c and k, we form E_2 by translating the whole dilate of E_1 to the right by $\frac{k}{d} = \frac{1}{d}$, thus taking $E_2 = [\frac{c}{d} - \frac{1}{2d^2} + \frac{1}{d}, \frac{c}{d} + \frac{1}{2d^2} + \frac{1}{d})$. We then form E_3 by splitting the dilate of E_2 in half and translating the left half by $-\frac{1}{d}$. This makes E_3 the same set as in the first example, so that its two halves match up with the two ends of E_1. Continuing the construction by giving T' the same definition on the rest of the odd E_i as on E_1, and the same definition on the rest of the even E_i as on E_2, we get a two interval scaling set:

$$E = \left[-\frac{1}{d+1}, \frac{1}{d^2-1}\right) \cup \left[\frac{1}{d+1}, \frac{d}{d^2-1}\right). \tag{1.10}$$

Unlike the previous example, this construction does not use translation by $\frac{l}{d}$, and thus does not require that d be an integer. However, in order to avoid overlap in our intervals (which would violate the requirement that T' be one-to-one), we need $d \geq 2$. The resulting wavelet set, for any real dilation $d \geq 2$, is given by

$$W = \left[-\frac{d}{d+1}, -\frac{1}{d+1}\right) \cup \left[\frac{1}{d^2-1}, \frac{1}{d+1}\right) \cup \left[\frac{d}{d+1}, \frac{d^2}{d^2-1}\right). \tag{1.11}$$

The family of wavelet sets (1.11) appears as an example in [13].

Other wavelet sets can be built by this same technique if we ask for different adjacencies. For example, for dilation by 2, if we ask that the second dilate of the halves of E_2 be adjacent to E_1 instead of the first dilate being adjacent, we get the Journé wavelet set $W = [-\frac{16}{7}, -2) \cup [-\frac{1}{2}, -\frac{2}{7}) \cup [\frac{2}{7}, \frac{1}{2}) \cup [2, \frac{16}{7})$. The same construction can be used to produce analogs of Journé for any dilation d such that $d^2 \in \mathbb{Z}$. If we alter the Journé construction by translating the whole dilate of E_1 to the right to form E_2, as we did before to construct scaling set (1.10), but still requiring the halves of the second dilate of E_2 match up with E_1, we can build a wavelet set for any real dilation $d > \frac{3}{2}$. For $d = 2$, the resulting wavelet set is $W = [-\frac{4}{7}, -\frac{2}{7}) \cup [\frac{2}{7}, \frac{3}{7}) \cup [\frac{12}{7}, \frac{16}{7})$. We can generalize both example (1.10) and this one-sided Journé-like construction by asking that the j^{th} dilate of the halves of E_2 be adjacent to $E_!$, where $j \geq 1$. (Here, $j = 1$ corresponds to scaling set (1.10) and $j = 2$ corresponsd to the one-sided Journé.) There are many solutions for c and k in the analogous equations to (1.7); one consistent with our previous examples is

$$k = j, \qquad c = \frac{2j - d^j}{2(d^{j+1} - 1)}. \tag{1.12}$$

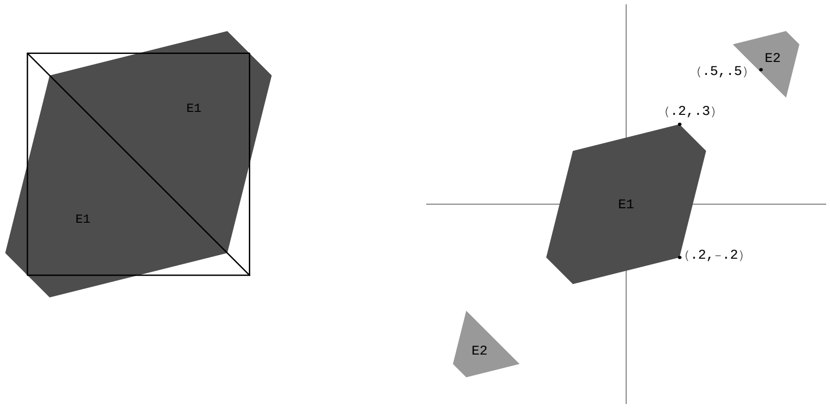
The resulting scaling set E has $j+1$ pieces, with the largest (formed from E_1) given by $[\frac{j-d^j}{d^{j+1}-1}, \frac{j}{d^{j+1}-1})$, the second given by the dilate of the first translated by $\frac{j}{d}$, and the rest given by successive dilates of the second. This generalized example can be built for any dilation $d \geq \max(j^{\frac{1}{j}}, \frac{j+1}{j})$. (The first restriction is required to ensure that the origin is in E_1 and the second to avoid overlap.) Thus by taking j large enough, we can get a wavelet set consisting of a finite number of intervals for any real dilation $d > 1$. Other one-dimensional examples can be found in [3].

We will now use these one-dimensional constructions as a starting point to build the two dimensional examples. As before, we will make our first example by splitting E_1 at the center, and forcing the images of this center in E_3 to match up with the edges of E_1. We can make the positions match up by using the same values for c, l, k as in the first one-dimensional construction, here used in both horizontal and vertical directions. In two dimensions, however, we have the additional complication of needing the size and shape to match as well. We will accomplish this by making E_1 a truncated diamond with axis along the line $y = x$. We think of the truncated diamond as the union of two trapezoids, with the longer of two parallel sides coinciding at the center of E_1, and the shorter at the edges with length $\frac{1}{d^2}$ times the length of the center line. (See Figure 1.2(a) below.) The fact that slopes are preserved by integer dilations is essential to making this construction work. For the sake of clarity, we describe the details of this construction first for the specific case $d = 2$.

Recall that E_1 must satisfy equation 1.4, and so must consist of translates by $\mathbb{Z}^2/2$ of the points in the dilated square $[-\frac{1}{4}, \frac{1}{4})^2$. Because E_1 had a center value of $c = 0$ in the one-dimensional example for dilation by 2, we do not need to move the dilated square before forming the correct shape in this case. To make the truncated diamond, we remove triangles from the corners of the dilated square in the second and fourth quadrants, and translate these triangles to positions adjacent to the corners of the square in the first and third quadrants. We choose the size of the triangle to make the outer edges of the truncated diamond have length $\frac{1}{4}$ times the length of the centerline. Specifically, we translate the triangle in the fourth quadrant with vertices $(\frac{1}{4}, -\frac{1}{4})$, $(\frac{2}{10}, -\frac{2}{10})$, and $(0, \frac{1}{4})$ up by $(0, \frac{1}{2})$ and the one with vertices at $(\frac{1}{4}, -\frac{1}{4})$, $(\frac{2}{10}, -\frac{2}{10})$, and $(\frac{1}{4}, 0)$ to the left by $(-\frac{1}{2}, 0)$. Symmetrically defined translations of triangles in the second quadrant complete the construction of the truncated diamond E_1. (See Figure 1.2(a).) Then, to form E_2, we first dilate E_1 to $\frac{1}{2}E_1$ and then translate the dilated upper right trapezoid by $(\frac{1}{2}, \frac{1}{2})$ and the dilated lower left trapezoid by $(-\frac{1}{2}, -\frac{1}{2})$. (See Figure 1.2(b).)

It is now easy to see that by taking E_3 to be $\frac{1}{2}E_2$, we can make E_3 be just an extension of the truncated diamond E_1. For the construction of E_i, $i > 3$, we define T' following the first one-dimensional example. That is, for $x \in E_i$ with i odd, we let $T'((x, y)) = (\frac{x}{2} + \frac{1}{2}, \frac{y}{2} + \frac{1}{2})$ for $y \geq -x$ and $T'((x, y)) = (\frac{x}{2} - \frac{1}{2}, \frac{y}{2} - \frac{1}{2})$ for $y < -x$. For even i, T' takes $(x, y) \in E_i$ to $(\frac{x}{2}, \frac{y}{2})$, as is required by the rules of the construction technique. The resulting scaling set E, and its wavelet set $W = 2E \setminus E$ are shown below in Figure 1.3 .

Variations on this construction as described for the one-dimensional case will work here as well. If we leave all the odd E_i as in the previous construction but put both halves of the even $E_i's$ in the first quadrant, we get a scaling set analogous to the 1-dimensional (1.10), which results in the wavelet set shown in Figure 1.4a. If instead we form E_2 by translating the upper right trapezoid of E_1 to the lower left, and the lower left trapezoid to the upper right, we get the "connected" wavelet set shown in Figure 1.4b. Notice that these alternative generalized scaling sets could also be formed directly from the first by translating pieces of E by lattice elements in such a way that the containment $E \subset 2E$ is maintained.



(a) (b)

Fig. 1.2. BuildingE_1 from $[-\frac{1}{4}, \frac{1}{4})^2$ and E_2 from E_1

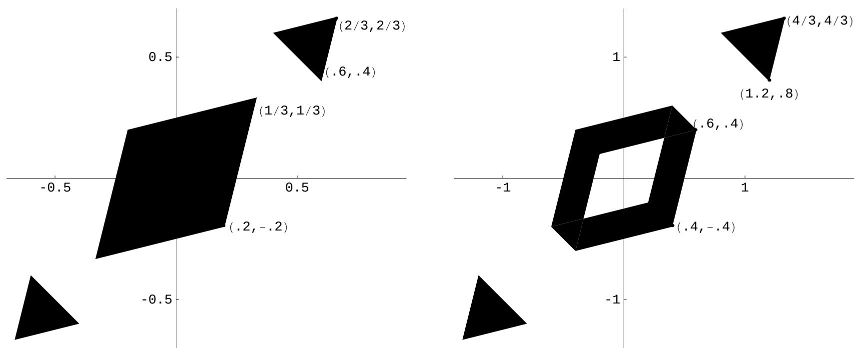
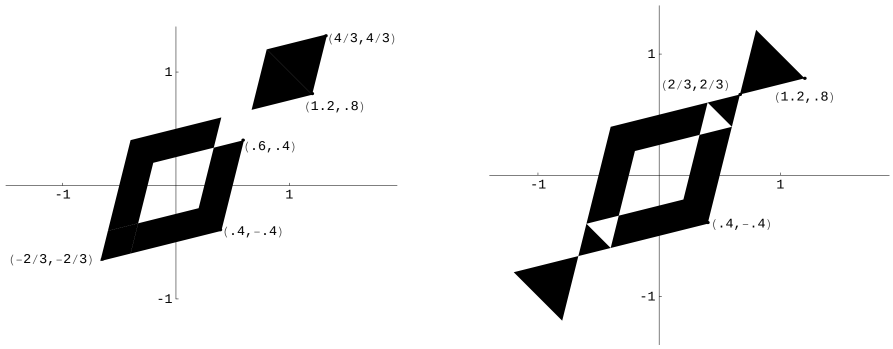


Fig. 1.3. A generalized scaling set and the corresponding wavelet set for dilation by 2

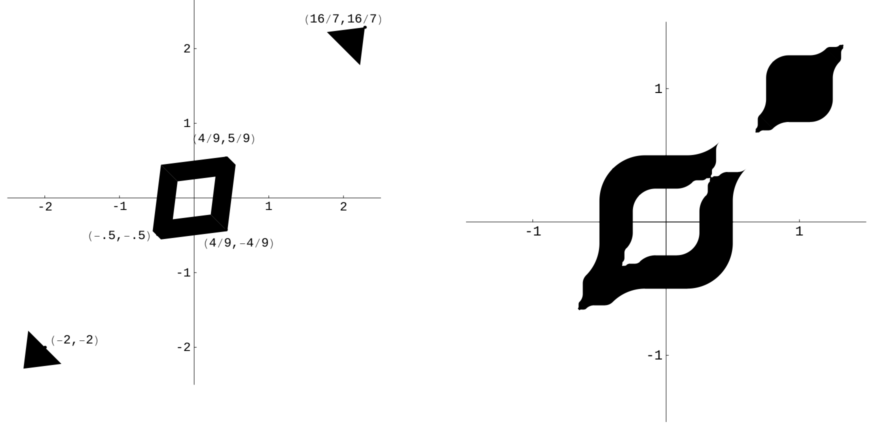
We give two final variations that cannot easily be seen as resulting from translates of the original E. If we form E_2 by translating the two dilated trapezoids of E_1 out by $\pm(1,1)$, then dilate them twice to get adjacency to E_1, the resulting 2-dimensional wavelet set is analogous to Journé. (See Figure 1.5a.) This provides an example of a wavelet set in $\mathbb{R}^2$ whose multiplicity function (dimension function) takes on values greater than 1. Finally, if we build E_1 from the square $[-\frac{1}{4}, -\frac{1}{4})^2$ by translating rounded triangles, and then complete E as for Figure 1.4a, we get the rounded wavelet set pictured in Figure 1.5b. Unlike the previous 2-dimensional examples in this section, Example 1.5b is not the finite union of convex sets, but it does have the property that its boundary is smooth except at a finite number of points. Many other variations are possible, including the obvious rotations of the given examples by multiples of $\frac{\pi}{2}$.

Now we describe briefly how to construct analogous two dimensional examples for any scalar dilation d. Just as in the one-dimensional example, we note that any square of the form $[c - \frac{1}{2d}, c + \frac{1}{2d})^2$ with $-\frac{1}{2d} < c < \frac{1}{2d}$ will contain a neighborhood of the origin and satisfy Equation 1.4. So, to generalize Example 1.3 or Example 1.4, we first locate the center of square as in the one-dimensional



(a) (b)

Fig. 1.4. A two-piece wavelet set and a connected wavelet set for dilation by 2



(a) (b)

Fig. 1.5. A wavelet set analogous to Journé and a rounded wavelet set for dilation by 2

example to be $(c, c) = (\frac{2-d}{2d^2-2}, \frac{2-d}{2d^2-2})$. To form E_1, we then modify the square as we did for dilation by 2, to form a truncated diamond whose cutoff edges are $\frac{1}{d^2}$ as long as the parallel centerline. Again following the one-dimensional example, we let E_2 be the set that results from translating the dilate of the upper right trapezoid by $(\frac{1}{d}, \frac{1}{d})$ and the lower left by $(-\frac{d-1}{d}, -\frac{d-1}{d})$. The set $E_3 = \frac{1}{d}E_2$ then exactly matches up with the outer edge of E_1. We finish the construction as before by letting T' have the same definition on all odd and even E_i as on E_1 and E_2 respectively. The wavelet set for dilation by the integer $d \geq 2$ can then be formed by $W = dE \setminus E$. The other forms discussed for dilation by 2 above can also be built for dilation by an arbitrary integer $d \geq 2$. For example, Figure 1.6 shows the one sided wavelet set for dilation by 3.

Just as in the 1-dimensional constructions, the one-sided forms 1.4a and 1.5b can also be built for any scalar dilation $d \geq 2$; the Journé-like 1.5a can be build for any scalar dilation d such that $d^2 \in \mathbb{Z}$. Wavelet sets for any other scalar dilation $d \geq 1$ can be obtained using the one-sided generalized Journé with $(j+1)$-piece scaling sets. We use the values of k and c given in Equation

1.12, and build the original truncated diamond E_1 so that the lengths of the truncated edges are $\frac{1}{d^{j+1}}$ as long as the center line. As in the 1-dimensional case, this will yield a wavelet set for any $d \geq \max(j^{\frac{1}{j}}, \frac{j+1}{j})$, so that by taking j large enough, we can build a wavelet set that is the finite union of polygons for any real scalar dilation $d > 1$.

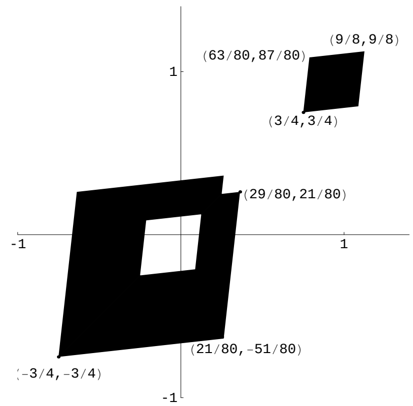


Fig. 1.6. A two-piece wavelet set for dilation by 3

Theorem 2 guarantees that all of the examples constructed in this section are wavelet sets. It is interesting to note that therefore they all necessarily satisfy Theorem 1. Figure 1.7 below shows this property for the dilation by 3 wavelet set of Figure 1.6.

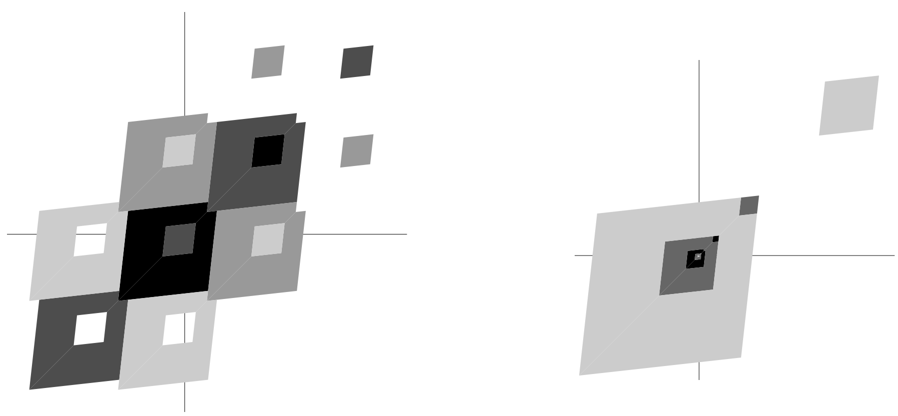


Fig. 1.7. Tiling $\mathbb{R}^2$ by translation and dilation with dilation 3 wavelet set

1.4 Generalizations

The 2-dimensional examples built in the previous section depend on picking the maps T and T' so that one of the sets E_i, $i \geq 2$ matches up nicely with the outer edges of E_1. This match up depends on the fact that scalar dilations preserve slopes. Thus, the techniques used in Section 1.3 cannot easily be generalized to other expansive matrix dilations.

The one type of non-scalar dilation that can be seen by previous work to have a simple wavelet set is any expansive integer matrix with determinant of absolute value 2. This class of matrices had already been marked as a special case since they are the only dilations in dimension 2 that can give MRA wavelets. Such matrices can be grouped into six classes, and then representatives of each class can be shown to have simple wavelet sets as follows. Two matrices A and B are *integrally similar* if there is an integer matrix C of determinant ± 1 such that $A = CBC^{-1}$. It is easy to see that if A has a wavelet set that is the union of a finite number of polygons, then any integrally similar matrix B does as well. All deteminant ± 2 matrices are integrally similar to one of the following 6 matrices: $\begin{pmatrix} 0 & \pm 2 \\ 1 & 0 \end{pmatrix}$, $\pm \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\pm \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$ (see [11], [24], [25]). Simple wavelet sets can be found for the first pair of matrices (Figure 1.8(a)) using tensor products of one-dimensional wavelets and scaling functions (see, e.g. [35]). Calogero [12] and Gu and Han [20] independently produced a simple wavelet set for the quinconx matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (Figure 1.8(b)) that is symmetric with respect to the origin, and thus a wavelet set for $-\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ as well. Finally, Bownik and Speegle [11], found a simple wavelet set for $\pm \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$ (Figure 1.8(c)), thus showing that simple wavelet sets do exist for all matrices with determinant of absolute value 2.

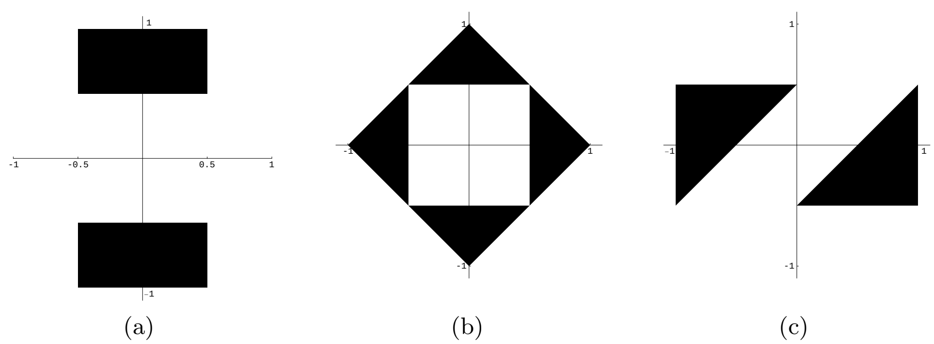


Fig. 1.8. Wavelet sets for determinant 2 matrices

We know that the construction technique described in Section 1.2 can be used to build these determinant ± 2 wavelet sets, since that technique was shown in [3] to produce all wavelet sets. However, our construction of the determinant ± 2 wavelet sets depends on a lucky guess to find the maps T and T', rather than a general procedure for choosing these maps, such as the one described in Section 1.3. Thus, the determinant ± 2 wavelet sets are currently hard to generalize using our procedure. Since many different pairs of maps T and T' can be used to build the same wavelet set, it is possible that a more general approach to the determinant ± 2 sets will be found. For now, it remains an open question whether simple wavelet sets (either the finite union of polygons or even the finite union of convex sets) exist for arbitrary expansive integer-valued matrix dilations in $\mathbb{R}^2$. One negative result along these lines is known for dilation by a non-expansive, non-integer valued 2×2 matrix. Darrin Speegle showed in 2003 [34] that the matrix $\begin{pmatrix} 2 & 0 \\ \sqrt{2} & 1 \end{pmatrix}$ does have wavelet sets, but that any wavelet set must have empty interior. For dimensions higher than 2, the question is almost entirely open. For example, it is unknown whether a wavelet set for any scalar dilation in $\mathbb{R}^3$ can be a finite union of polyhedra or even a finite union of convex sets.

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