

Smooth well-localized Parseval wavelets based on wavelet sets in \mathbb{R}^2

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ABSTRACT. A generalized filter construction is used to build non-MRA Parseval wavelets for dilation by 2 in $L^2(\mathbb{R}^2)$. These examples have the same multiplicity functions as wavelet sets, yet can be made to be C^r with C^r Fourier transform for any fixed positive integer r .

1. Introduction

Wavelets in $L^2(\mathbb{R}^2)$ use translates and dilates of a single function to provide bases or frames that are particularly useful in applications such as image processing. Here, by an *orthonormal wavelet*, we will mean a function $\psi \in L^2(\mathbb{R}^2)$ such that $\{\psi_{j,k} \equiv 2^j \psi(2^j x - k)\}$ form an orthonormal basis for $L^2(\mathbb{R}^2)$. The function ψ is instead called a *Parseval wavelet* if $\{\psi_{j,k}\}$ form a Parseval frame (normalized tight frame) for $L^2(\mathbb{R}^2)$, so that for each $f \in L^2$ we have $\|f\|^2 = \sum_{j,k} |\langle f, \psi_{j,k} \rangle|^2$. In either case, the dilates and translates of the wavelet can be used to reconstruct all L^2 functions.

The earliest wavelets in two-dimensional space were built using tensor products of their one-dimensional counterparts, and thus consisted of four wavelet functions rather than a single function as in the definition above. Dai, Larson and Speegle [13] showed in 1997 that even in a much broader context, single wavelets exist in the form of wavelet set wavelets, whose Fourier transform $\hat{\psi}$ is the characteristic function of a set. Because of their discontinuous Fourier transform, wavelet set wavelets are not well-localized, and thus while important to wavelet theory, they are not very useful in applications. In fact, any single orthonormal wavelet for dilation by 2 in $L^2(\mathbb{R}^2)$ must be non-MRA [3] and thus cannot be simultaneously smooth and well-localized [15]. Parseval wavelets are not so restricted.

In this paper we use generalized filters on wavelet sets in \mathbb{R}^2 to build single Parseval wavelets that are both smooth and well-localized. Unlike other such 2-dimensional Parseval wavelets in the literature (see e.g. [1]), those produced in this paper have known multiplicity functions, filters and generalized multiresolution structures. In Section 2 we describe the use of generalized filters to build these smooth, well-localized Parseval wavelets on any wavelet set that has a relatively simple structure. In Section 3, we use this procedure to build such wavelets on

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the wedding cake wavelet set of Dai, Larson and Speegle [14] and on the recently discovered diamond wavelet set of [16], which is the finite union of convex polygons.

2. Generalized filters on wavelet sets

The generalized filters we will use to build our wavelets are based on a multi-resolution structure that we now describe. Any orthonormal wavelet ψ can be used to divide $L^2(\mathbb{R}^2)$ into a nested sequence of closed subspaces defined by the level of zoom: $V_j = \overline{\text{span}}\{\psi_{i,k} : i < j\}$. If the $\{V_j\}$ come from an orthonormal wavelet, they form a **Generalized Multiresolution Analysis (GMRA)** [7], defined by the properties:

- (1) $V_j \subset V_{j+1}$
- (2) $V_{j+1} = \{\delta(f) \equiv 2f(2x)\}_{f \in V_j}$
- (3) $\cup V_j$ dense in $L^2(\mathbb{R}^2)$ and $\cap V_j = \{0\}$
- (4) V_0 is invariant under translation by \mathbb{Z}^2 .

This definition differs from the classical multiresolution analysis (MRA) only in condition (4): An MRA requires that V_0 has a scaling function ϕ such that translates of ϕ form an orthonormal basis for V_0 , while a GMRA requires only that V_0 be invariant under translation by the integers. In spite of this difference, it is shown in [7] that a GMRA has almost as much structure as an MRA. Translation is a unitary representation of \mathbb{Z}^2 on V_0 , and thus is completely determined by a multiplicity function $m : \mathbb{R}^2/\mathbb{Z}^2 \mapsto \{0, 1, 2, \dots, \infty\}$ describing how many times each character occurs as a subrepresentation. The multiplicity function is identically 1 if and only if the GMRA is actually an MRA. Further, by writing $V_1 = V_0 \oplus W_0$, representation theory can be used (see [7]) to show that the GMRA has an associated orthonormal wavelet if and only if the multiplicity function satisfies a *consistency equation*:

$$(2.1) \quad m(\zeta) + 1 = \sum_{l=1}^4 m\left(\frac{\zeta}{2} + \omega_l\right),$$

where we parameterize $\mathbb{R}^2/\mathbb{Z}^2$ by $[-\frac{1}{2}, \frac{1}{2})^2$, and denote the preimages of 0 under multiplication by 2 mod 1 by $\{\omega_l\} = \{0, (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$. The multiplicity function for the wavelet determined by a wavelet set W is given by

$$(2.2) \quad m(\zeta) = \sum_{k \in \mathbb{Z}^2} \chi_E(\zeta + k),$$

where $E = \cup_{j < 0} 2^j W$ is the generalized scaling set that determines the GMRA via $\widehat{V}_j = L^2(2^j E)$ [7]. In this paper, we will use the multiplicity functions of known wavelet sets to build smooth and well-localized Parseval wavelets with the same multiplicity function.

To simplify our construction, we will restrict our attention to wavelet sets whose multiplicity functions are bounded by 1 and supported on sets with relatively simple geometric structure. As is traditional in the literature, we label that support $S_1 = \{\zeta : m(\zeta) \geq 1\}$. Note that S_1 is the same as the generalized scaling set E mod 1. Since a single wavelet in $L^2(\mathbb{R}^2)$ cannot be an MRA wavelet, we know that S_1 is a proper subset of $[-\frac{1}{2}, \frac{1}{2})^2$. Our particular restrictions on S_1 are as follows. Our *first requirement* is that S_1 be oriented along either the x axis, the y axis, or one of the lines $y = \pm x$, which we call the *central axis* of S_1 . In particular, we require that $S_1 \subset A$, where A is one of the sets $\{|x| \leq \frac{1}{4}\}$, $\{|y| \leq \frac{1}{4}\}$, $\{|x - y| \leq \frac{1}{4}\}$,

or $\{|x + y| \leq \frac{1}{4}\} \pmod{1}$. The existence of such an axis is usually apparent in the wavelet set $W = 2E \setminus E$. Our *second requirement* is that all the connected components $\pmod{1}$ of the interior S_1° are star-shaped, and that the origin is a star center of one of the components. Because of the consistency equation (2.1), for each $\zeta \in S_1$, there will be exactly two of the four preimages under dilation by 2, $\{\frac{\zeta}{2} + \omega_l\}$, that are inside S_1 . Our *third requirement* is that the collection of the two preimages of the points in a given component of S_1° has itself two connected components. We let $P \subset S_1$ be the set of these two preimages of all the points in S_1 . For points in $C \equiv$ the connected component of 0 in S_1 , one of these preimages will be $\frac{1}{2}C$, and the other will be a component around one of the $\omega_l \in \{(0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\}$ (depending on the central axis). In order to use this point to describe the central axis, we will further differentiate between $\omega_l = (\frac{1}{2}, \frac{1}{2})$ and $\omega_l = (\frac{1}{2}, -\frac{1}{2})$, based on the location of the preimages of ω_l inside S_1 , which will distinguish between central axes $y = x$ and $y = -x$. We let $\omega_l^* \in \{(0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\}$ be the point that with 0 determines the central axis of S_1 . Our *fourth requirement* is that $\omega_l^* \notin C$ but $\pm\frac{4}{9}\omega_l^* \in C$.

The final two requirements on S_1 have to do with its boundary. Wavelet sets and their multiplicity functions are only determined a.e.. However, for convenience in constructing our smooth filters, we take S_1 and P to include the boundaries of their interiors, which we label as ∂S_1 and ∂P respectively. Our *fifth requirement* is that ∂S_1 be composed of a countable (possibly finite) number of line segments whose endpoints have at most a finite number of limit points. Because of the existence of a central axis, for ζ in a small neighborhood of a boundary point of S_1 , only a fixed two terms can be nonzero in the right-hand side of the consistency equation, so that points on the boundary of S_1° must satisfy the consistency equation as well. Because of this and the third requirement, ∂P will be a subset of the two preimages of ∂S_1 , and thus will also satisfy the property of the fifth requirement. The consistency equation implies that for each ζ on the boundary of $S_1 \pmod{1}$, exactly one of the preimages lies on the boundary of S_1 as well, and the other lies in the interior of S_1 . Since all boundary points of P are preimages of boundary points of S_1 , this divides ∂P into two types of points and insures that each point on the boundary of S_1 has one preimage of each type. Our final, *sixth requirement* of the support of the multiplicity function is that there are only a finite number of *transition points* between these two types of boundary points in ∂P , and that none of these transition points coincide with limit points of the line segment endpoints and none lie on the central axis. As a consequence, there exists an ϵ_1 neighborhood of the central axis that contains no transition points. We say that a multiplicity function has *star-simple single-axis support* if all of six of these requirements are met.

Many of the examples of dilation 2 wavelet sets in \mathbb{R}^2 that appear in the literature have multiplicity functions with star-simple single-axis support. Examples of this type include the wedding cake set of [14], the alternative wedding cake and wedding night sets of [12], and the diamond sets of [16] (except for the Journé-like example that has points with multiplicity 2). The technique developed in this paper can be modified to work on some other examples that fail to have star-simple single-axis support, such as the pine tree set of [19] (which fails the third requirement, but only for one component of S_1°), and the four corners set of [14] and [17] (which has two axes instead of one). However, examples such as these introduce

complications that must be dealt with on a case by case basis. Other examples, such as the windmill set of [7] and [11], as well as the "origin as a limit point" set of [17], fail to have star-simple support for more serious reasons (more than one component around a preimage of the origin), which cannot be easily overcome.

To build our Parseval wavelets with the same multiplicity functions as wavelet sets with the simple geometric structure described above, we use a generalization of MRA filter techniques to GMRA's, developed in [4] and [5]. For any GMRA, there exists a unitary equivalence between translation on V_0 and multiplication by exponentials on $\bigoplus L^2(S_j)$, where $S_j = \{\zeta : m(\zeta) \geq j\}$, which plays a similar role to that of the classical Fourier transform in the MRA case. Generalized filters associated to the multiplicity function m can be built using this unitary equivalence. For the multiplicity function for a wavelet set with $m \leq 1$, the generalized filters are two periodic functions, h and g , which are supported on the periodization of $S_1 = \{\zeta : m(\zeta) \geq 1\}$, and satisfy orthonormality conditions similar to those of classical filters. In particular, we will use the following special case of a theorem from [5].

THEOREM 2.1. *Given a multiplicity function m for a GMRA that satisfies $m \leq 1$, suppose h and g are periodic functions that are supported on the periodization of $S_1 = \{\zeta : m(\zeta) \geq 1\}$, Lipschitz continuous in a neighborhood of the origin, and that satisfy the three generalized orthonormality conditions*

$$(2.3) \quad \sum_{l=1}^4 |h(\frac{\zeta}{2} + \omega_l)|^2 = 4\chi_{S_1}(\zeta)$$

$$(2.4) \quad \sum_{l=1}^4 |g(\frac{\zeta}{2} + \omega_l)|^2 = 4$$

and

$$(2.5) \quad \sum_{l=1}^4 h(\frac{\zeta}{2} + \omega_l) \overline{g(\frac{\zeta}{2} + \omega_l)} = 0,$$

where $\{\omega_l\} = \{(0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$ are the preimages of 0 under multiplication by 2 mod 1. Suppose in addition that h satisfies the generalized lowpass condition $|h(0)| = 2$. Let $\widehat{\phi}(\zeta) = \prod_{k=1}^{\infty} \frac{1}{2} h(\frac{\zeta}{2^k})$. Then the translates of ϕ determine the core subspace V_0 of a GMRA, and $\widehat{\psi}(\zeta) \equiv \frac{1}{2} g(\frac{\zeta}{2}) \widehat{\phi}(\frac{\zeta}{2})$ is the Fourier transform of a Parseval wavelet on $L^2(\mathbb{R}^2)$.

To design filters that we can use in this theorem to construct our C^r wavelets, we will first build functions h_r and g_r that have desirable properties on $[-\frac{1}{2}, \frac{1}{2}]^2$, and then take their periodizations. To satisfy the conditions of Theorem 2.1, we must take $h_r = g_r = 0$ on S_1^c . We also must take $h_r = 0$ and $|g_r| = 2$ on $S_1 \setminus P$. (The necessity of $|g_r| = 2$ on this set follows from (2.4) since by the consistency equation none of the other terms that appear on its left-hand side can be nonzero.) Note that thus g_r will necessarily be discontinuous on the boundary between $S_1 \setminus P$ and S_1^c . Since S_1 cannot be all of $[-\frac{1}{2}, \frac{1}{2}]^2$, equation (2.3) implies that h_r must have points of discontinuity as well. Our goal in the next lemma is to build filters h_r and g_r that are as close to C^∞ as possible, and that vanish rapidly near the points ω_l^* and $\frac{1}{3}\omega_l^*$. This second characteristic will be needed later to force the

infinite product in Theorem 2.1 to vanish rapidly at ∞ . Recall that P is the union of the two preimages of S_1° in S_1 under dilation by 2 mod 1, C is the connected component of the origin in S_1° , and ω_l^* is the nonzero preimage of 0, which together with 0, determines the central axis of S_1 .

LEMMA 2.2. *Let m be a multiplicity function that is bounded by 1 and has star-simple single-axis support. Then for each integer $r \geq 1$, there exist filters h_r and g_r on $[-\frac{1}{2}, \frac{1}{2}]^2$ whose periodizations satisfy the conditions of Theorem 2.1 and which have the following properties:*

- (1) $h_r = 0$ on $\partial P \cap S_1^\circ$ and h_r is nonzero a.e. on P .
- (2) h_r and g_r are C^∞ on $[-\frac{1}{2}, \frac{1}{2}]^2 \setminus \partial S_1$. Their partial derivatives of all orders are bounded on closed sets that contain no Transition Points, are 0 at $\zeta = 0$ and $\zeta = \omega_l^*$, and approach 0 as $\zeta \rightarrow (\partial P \setminus \{\text{Transition Points}\})$.
- (3) \exists an $\epsilon_2 > 0$ such that $|h_r(\zeta)| < |\zeta - \omega_l^*|^{r+2}$ and $|Dh_r(\zeta)| < |\zeta - \omega_l^*|^2$ for $|\zeta - t(\omega_l^*)| < \epsilon_2$, $\frac{8}{9} < t < \frac{10}{9}$, and D any partial differentiation operator of order less than or equal r .
- (4) \exists an $\epsilon_3 > 0$ such that $|h_r(\zeta)| < |\zeta \pm \frac{1}{3}\omega_l^*|^{r+2}$ and $|Dh_r(\zeta)| < |\zeta \pm \frac{1}{3}\omega_l^*|^2$ for $|\zeta \pm t(\omega_l^*)| < \epsilon_3$, $\frac{2}{9} < t < \frac{4}{9}$, and D any partial differentiation operator of order less than or equal r .

PROOF. Theorem 2.1 requires the support of h_r and g_r be contained in S_1 . As mentioned above, we are also required by the generalized orthonormality conditions to define $h_r = 0$ on $S_1 \setminus P$. We further define $h_r = 0$ on $\partial P \cap S_1^\circ$, as specified in the statement of the Lemma, and take $h_r = 2$ on $(\partial P \cap \partial S_1)$. Note that this definition is consistent with the orthonormality conditions, because requirement 3 of star-simple single-axis support requires that $2(\partial P) \subset \partial S_1$, and the consistency equation implies that for each ζ on the boundary of S_1 , exactly one of its preimages in P lies on the boundary of S_1 as well, and the other lies in the interior of S_1 . To finish the construction of h_r , we must define it on P to satisfy the orthonormality condition (2.3) as well as the Lemma's conditions (2), (3), and (4). We will first define a preliminary version \tilde{h}_r of h_r that has the same boundary values and satisfies the orthonormality condition (2.3) as well as conditions (1)-(3). For this, we use polar coordinates on each connected component of P in terms of a star center. That is, if R is a connected component of P with star-center c , we will use polar coordinates to define $\tilde{h}_r(\zeta - c)$ for $\zeta \in R$. After using this technique to define \tilde{h}_r on all connected components of P , we will alter \tilde{h}_r to form our filter h_r , which will also satisfy condition (4). Finally, we will use that definition to construct g_r in a manner similar to the classical derivation of high-pass filters from their low-pass counterparts, so that it satisfies conditions 2.4 and 2.5 as well as the conditions of the Lemma.

Let $f_{r,\alpha,\beta}$ be a monotonic C^∞ function on $[\alpha, \beta]$ such that $f_{r,\alpha,\beta}(\alpha) = 0$, $f_{r,\alpha,\beta}(\beta) = 1$, $|\frac{d^r}{dx^r} f_{r,\alpha,\beta}(x)| < |x - \alpha|^2$ for $|x - \alpha| < \frac{2}{3}|\beta - \alpha|$, and the one-sided derivatives of all orders of $f_{r,\alpha,\beta}(\alpha)$ vanish at both α and β . By the Mean Value Theorem, we know that $|f_{r,\alpha,\beta}(x)| < |x - \alpha|^{r+2}$ for $|x - \alpha| < \frac{2}{3}|\beta - \alpha|$. We will first use the $f_{r,\alpha,\beta}$ to define our preliminary filter \tilde{h}_r on $R =$ the connected component of ω_l^* in P , which we know (by the central axis requirement and by $\omega_l^* \notin C$) to be given by $R = \omega_l^* + \frac{1}{2}C$. Note that since $\frac{1}{2}C \subset C$, all of the boundary points of $\frac{1}{2}C$ have $\tilde{h}_r = 0$, and thus all of the boundary points of this region R have $\tilde{h}_r = 2$. As mentioned above, we will use polar coordinates on $R - c$, where the star-center c

can be taken to be ω_l^* in this case. Suppose the polar functions $\rho = a_j(\theta)$ defined on $\theta_j < \theta < \theta_{j+1}$ describe successive line segments in the boundary of $R - c$. (If the line segments in the boundary of R have more than one limit point, we will need more than one such sequence.) Our basic building block will be the function $2f_{r,0,1}(\frac{\rho}{a_j(\theta)})$ defined on the polar sector $(R - c) \cap \{\theta_j \leq \theta \leq \theta_{j+1}\}$. This function has the correct value on the boundary of P , and goes to 0 faster than ρ^{r+2} at 0. In order to accomplish a C^∞ transition between sectors, we use another copy of $f_{1,\alpha,\beta}$ to phase the boundary for each sector in during the last half of the previous sector, and out during the first half of the following sector. Specifically we define

$$(2.6) \quad \tilde{h}_r(\zeta - c) = \tilde{h}_r(\rho, \theta) = \begin{cases} 2f_{r,0,1}(\frac{\rho}{a_j(\theta)}) \left(f_{1, \frac{\theta_j + \theta_{j+1}}{2}, \theta_j}(\theta) \left(f_{r,0,1}(\frac{\rho}{a_{j-1}(\theta)}) - 1 \right) + 1 \right) & \text{if } \theta_j \leq \theta \leq \frac{\theta_j + \theta_{j+1}}{2} \\ 2f_{r,0,1}(\frac{\rho}{a_j(\theta)}) \left(f_{1, \frac{\theta_j + \theta_{j+1}}{2}, \theta_{j+1}}(\theta) \left(f_{r,0,1}(\frac{\rho}{a_{j+1}(\theta)}) - 1 \right) + 1 \right) & \text{if } \frac{\theta_j + \theta_{j+1}}{2} \leq \theta < \theta_{j+1}. \end{cases}$$

We then define \tilde{h}_r on $\frac{1}{2}C$ by

$$(2.7) \quad \tilde{h}_r(\zeta) = \sqrt{4 - (\tilde{h}_r(\zeta + \omega_l^*))^2}.$$

This definition is required to satisfy 2.3, since by the central axis requirement, the point $\zeta + \omega_l^*$ is the other preimage in P (under dilation by 2) of the point 2ζ . Note that these definitions satisfy all the requirements of the Lemma on the two preimages of C except (4). In particular, since one of the restrictions of star-simple is that $\pm \frac{4}{9}\omega_l^* \in C$, we have that $\pm \frac{7}{9}\omega_l^*$ is in the component R with star-center ω_l^* , so that $t(\omega_l^*) \in R$ for $\frac{7}{9} < t < \frac{11}{9}$, establishing condition (3) because of the rapidly vanishing at the center property of \tilde{h}_r for $\frac{2}{3}$ of the distance from the center to the edge.

We use a similar technique on other components of P that have the property that they do not contain any transition points between types of boundary points with $h_r = 2$ and $h_r = 0$. These components will come in pairs, like the preimages of C . We alter the technique used there in two ways: we will start by defining \tilde{h}_r on the component with boundary values of 0, rather than on the one with boundary values of 2, and we scale the maximum value of \tilde{h}_r on these components, so that its derivative will be bounded even if we have an infinite number of components. Thus, let R is a component of P other than $\frac{1}{2}C$, all of whose boundary points have $h_r = 0$, and let c be a star-center of R . As before, suppose the polar functions $\rho = a_j(\theta)$ defined on $\theta_j < \theta < \theta_{j+1}$ describe successive line segments in the boundary of $R - c$. Let $s = \frac{\min\{|\zeta - c| : \zeta \in \partial R\}}{\min\{|\zeta| : \zeta \in \partial \frac{1}{2}C\}}$, and define

$$(2.8) \quad \tilde{h}_r(\zeta - c) = \tilde{h}_r(\rho, \theta) = \begin{cases} 2sf_{r,1,0}(\frac{\rho}{a_j(\theta)}) \left(f_{1, \frac{\theta_j + \theta_{j+1}}{2}, \theta_j}(\theta) \left(f_{r,1,0}(\frac{\rho}{a_{j-1}(\theta)}) - 1 \right) + 1 \right) & \text{if } \theta_j \leq \theta \leq \frac{\theta_j + \theta_{j+1}}{2} \\ 2sf_{r,1,0}(\frac{\rho}{a_j(\theta)}) \left(f_{1, \frac{\theta_j + \theta_{j+1}}{2}, \theta_{j+1}}(\theta) \left(f_{r,1,0}(\frac{\rho}{a_{j+1}(\theta)}) - 1 \right) + 1 \right) & \text{if } \frac{\theta_j + \theta_{j+1}}{2} \leq \theta < \theta_{j+1}. \end{cases}$$

On the corresponding components where all of the boundary points have $h_r = 2$, we again use (2.7) (as we must).

Finally, we define \tilde{h}_r on a component R of P° that contain both boundary points with $h_r = 0$ and boundary points with $h_r = 2$. Again, we will define these functions in polar coordinates with respect to a star-center c . We define $\tilde{h}_r = \sqrt{2}$ at this star center. On any sectors between the midpoint of a boundary line segment and its end, where values on the boundary segment are constantly 0 with no jump at the endpoint, we use (2.8) scaled by $s = \frac{1}{\sqrt{2}}$. On any such sectors where values on the boundary segment are constantly 2 (again with no discontinuity at the endpoint) we use (2.7). Now, let $\theta = q$ give a point of discontinuity of h on the boundary, and let the adjacent boundary component where $h = 0$ be given by $r = a_1(\theta)$ defined on $\theta_1 \leq \theta \leq q$. We will use our function $f_{1,\alpha,\beta}$ to make a transition between the value $\sqrt{2}$ near the center, and the required 0 at the boundary. The transition will begin at a distance $\rho = \rho_c(\theta)$ from the center, where ρ_c is $a_1(q)$ at $\theta = q$ and 0 at $\theta = \frac{\theta_1+q}{2}$. To make the transition point along a C^∞ curve, we again use $f_{1,\alpha,\beta}$.

$$(2.9) \quad \rho_c(\theta) = a_1(q)f_{1,\frac{\theta_1+q}{2},q}(\theta)$$

Using this as the transition point, we get the following formula for \tilde{h}_r :

$$(2.10) \quad \tilde{h}_r(\zeta - c) = \tilde{h}_r(\rho, \theta) = \begin{cases} \sqrt{2} & \text{if } \rho \leq \rho_c(\theta) \\ \sqrt{2}f_{1,a_1(\theta),\rho_c(\theta)}(\rho) & \text{if } \rho > \rho_c(\theta) \end{cases}$$

We define \tilde{h}_r on the $h = 2$ side of the discontinuities using (2.7). With this, we have a function \tilde{h}_r on all of S_1 that satisfies Theorem 2.1 as well as conditions (1)-(3) of the Lemma.

To finish the definition of h_r , we will alter \tilde{h}_r in such a way to satisfy condition (4) as well. To this end, we let $R = S_1 \cap \{\zeta : |\zeta - \frac{1}{3}\omega_l^*| < \frac{1}{12}\}$, and define a function $\gamma(\rho, \theta)$ on $R - c$ with $c = \frac{1}{3}\omega_l^*$ using (2.6). In this case, some of the boundary segments $\rho = a_j(\theta)$ may be segments of circles, but the construction works the same. We use an analogous definition on $R = S_1 \cap \{\zeta : |\zeta + \frac{1}{3}\omega_l^*| < \frac{1}{12}\}$. We then define h_r as follows:

$$(2.11) \quad h_r(\zeta) = \begin{cases} \frac{1}{2}\tilde{h}_r(\zeta)\gamma(\zeta) & \text{if } \zeta \in S_1 \cap \{\zeta : |\zeta \pm \frac{1}{3}\omega_l^*| < \frac{1}{12}\} \\ \sqrt{4 - (\frac{1}{2}\tilde{h}_r(\zeta - \omega_l^*)\gamma(\zeta - \omega_l^*))^2} & \text{if } \zeta \in S_1 \cap \{\zeta : |\zeta \pm \frac{1}{3}\omega_l^*| < \frac{1}{12}\} + \omega_l^* \\ \tilde{h}_r(\zeta) & \text{else} \end{cases}$$

Since we have $\pm\frac{2}{9}\omega_l^* \in C$, this alteration does satisfy condition (4) of the Lemma, while retaining conditions (1)-(3) and the orthonormality condition (2.3). We have h_r nonzero on P except for a set of measure 0 since \tilde{h}_r is 0 only on the boundary, and we have added only two zero points at $\pm\frac{1}{3}\omega_l^*$.

It remains to define the filter g_r on S_1 . To first define g_r on P , we note that because of the consistency equation and the central axis requirement, we have exactly two preimages of each point in S_1 that lie in P , and these two preimages have a constant difference of ω_l^* . Thus, the orthonormality condition (2.5) has two nonzero terms on the right hand side whose arguments differ by a constant, and therefore relates g_r to h_r in an analogous way to the classical one-dimensional filter equation. Thus we are able to use a modification of the classical definition of g_r

from h_r , that is

$$(2.12) \quad g_r(\zeta) = e^{\pi i \langle \zeta, \omega_i^* \rangle / \|\omega_i^*\|^2} \overline{h_r(\zeta + \omega_i^*)},$$

This definition will satisfy (2.4) and (2.5) exactly as the corresponding definition does in the classical case, and will satisfy the requirements of the Lemma because h_r does. To complete the definition of g_r , we extend the definition above to $S_1 \setminus P$ in a C^∞ manner, so that $|g_r| \equiv 2$ on $S_1 \setminus P$. \square

We now use the periodizations of these functions h_r and g_r together with Theorem 2.1 to construct our Parseval wavelet.

THEOREM 2.3. *Given any wavelet set for dilation by 2 in $L^2(\mathbb{R}^2)$ whose multiplicity function satisfies $m \leq 1$ and which has central-axis star-simple support, there exists a Parseval wavelet ψ_r , for each integer $r \geq 1$, with the same multiplicity function as the wavelet set, and such that $\psi_r \in C^r$ and $\zeta^r \psi_r(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$.*

PROOF. Let ϕ_r and ψ_r be defined from h_r and g_r by Theorem 2.1. As our first step, we will show that in spite of the discontinuities of h_r , we have that the infinite product $\hat{\phi}_r(\zeta) = \prod_{k=1}^{\infty} \frac{1}{2} h_r(\frac{\zeta}{2^k})$ is a C^∞ function for any $r \geq 1$. Fix an $r \geq 1$ and write $\hat{\phi}_r(\zeta) = T_1(\zeta)T_2(\zeta)$, where T_1 is a finite product of dilates of h_r , and the remaining infinite product T_2 is the same as $\hat{\phi}_r$ defined on a fixed neighborhood N of 0 that is inside $\frac{1}{2}C$. (If $\zeta \in N$, we take $T_1(\zeta) = 1$.) We will establish first that T_2 is C^∞ , and then that T_1 is as well.

Note that h_r has no discontinuities inside of N and thus is C^∞ with bounded partial derivatives of all orders there. Recall also, that by its construction in Lemma 2.2, there is a neighborhood of 0 on which h_r satisfies $|h_r(\zeta) - 2| < |\zeta|$, and also for each partial differentiation operator (of any order) D , there is a neighborhood of 0 such that $|Dh_r(\zeta)| < |\zeta|$. Thus we have that the sequence of partial products of T_2 , as well all the sequences of any given partial derivative of these partial products, are uniformly Cauchy, with the difference between the n^{th} and m^{th} elements bounded by a constant times $\sum_{j=n}^m \frac{1}{2^j}$. Thus each of these sequences converge uniformly to a continuous function, and $T_2 \in C^\infty$.

The infinite differentiability of T_1 follows from the fact that whenever h_r has a point p of discontinuity, that point is on the boundary of S_1 . Thus $\frac{p}{2}$ is either again on the boundary of S_1 , or on the boundary of P in the interior of S_1 . After a finite number of steps, the latter must be the case. Since h is 0 with all derivatives also 0 at such points, this factor cancels out the discontinuity of the previous terms. This is true even if p was a transition between the two types of boundary points, since by construction in Lemma 2.2, the size of the k^{th} order partial derivatives of h_r at a distance d from p is on the order of $\frac{1}{d^k}$. This is cancelled out by the next factor $h_r(\frac{p}{2})$, since by the third requirement of star-simple central-axis support, $\frac{p}{2}$ is in $\partial P \cap S_1^\circ$, where all partial derivatives approach 0 faster than any polynomial. Thus the function $\hat{\phi}_r \in C^\infty$. Note that the above argument also shows that $\hat{\phi}_r$ has value 0 and partial derivatives of all orders equal to 0 on the periodization of the boundary of S_1 . This implies that $\hat{\psi}_r$ is also C^∞ , since the only discontinuities of g_r occur at the boundary of S_1 , and these will be cancelled by multiplication by $\hat{\phi}_r$.

Next we show that $\zeta^{r+1} \hat{\phi}_r(\zeta) \rightarrow 0$ and $\zeta D(\hat{\phi}_r)(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$, where D represents a partial differentiation of any order less than or equal to r . It will

then follow that $\hat{\psi}_r$ will satisfy these same properties, and thus that $\psi_r \in C^r$ and $\zeta^r \psi_r(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$.

We first note that the central axis requirement on S_1 implies that support of $\hat{\phi}_r$ is contained in one of the four sets $\{(x, y) : |x| \leq \frac{1}{2}\}$, $\{(x, y) : |y| \leq \frac{1}{2}\}$, $\{(x, y) : |y - x| \leq \frac{1}{2}\}$ or $\{(x, y) : |y + x| \leq \frac{1}{2}\}$, depending on the central axis of S_1 . For example, in the case where the central axis is the y axis, the support of the periodization of h_r excludes the set $\{\frac{1}{4} < |x| \leq \frac{1}{2}\}$. But then $2^k \{\frac{1}{4} < |x| \leq \frac{1}{2}\}$ must be excluded from the support of $\hat{\phi}_r(\zeta) = \prod_{k=1}^{\infty} \frac{1}{2} h_r(\frac{\zeta}{2^k})$ for all $k \geq 1$, from which the claim follows for this case. The other cases are similar.

As a consequence, there exists a constant k_0 such that for any $\zeta \in \text{support}(\hat{\phi}_r)$, and for all $k > k_0$, the distance of $\frac{\zeta}{2^k}$ from the central axis is less than the minimum of ϵ_1 (in the definition of star-simple central-axis support), ϵ_2 and ϵ_3 (in Lemma 2.2), and .2 (to insure that a distance of $\frac{1}{18}$ from ω_l^* along the central axis will give no more than a distance of $\frac{1}{4}$ from ω_l^* to a corresponding point on the edge of the neighborhood). We call this neighborhood of the central axis N . As argued above, the unbounded derivatives of a factor of $h_r(p)$, for p a transition point, are cancelled out by the next factor $h_r(\frac{p}{2})$. Thus we know that partial derivatives of all orders of $\prod_{k=1}^{k_0+1} \frac{1}{2} h(\frac{\zeta}{2^k})$ are bounded, with a uniform bound depending on k_0 . Since $\prod_{k=1}^{k_0+1} \frac{1}{2} h(\frac{\zeta}{2^k})$ is also itself bounded by 1, without loss of generality, we may assume that ζ itself is in the neighborhood N . (We lose at most a fixed ratio 2^{k_0+1} in our estimate of the size of ζ , which becomes insignificant as $\zeta \rightarrow \infty$.)

Define $t(\zeta)$ by $|\zeta - t(\zeta)\omega_l^*| = \min\{|\zeta - t\omega_l^*| : t \in \mathbb{R}\}$. We divide $[-1, 1)$, the range of $t(\zeta)$ taken mod 2, into two regions. (We consider $t(\zeta)$ mod 2 rather than mod 1 since it defines the multiple of ω_l^* , which is a $\frac{1}{2}$ point.) Region 1 consists of the three intervals (mod 2) given by $([\frac{8}{9}, 1) \cup [-1, -\frac{8}{9}]) \cup [\frac{2}{9}, \frac{4}{9}) \cup [-\frac{4}{9}, -\frac{2}{9})$. When $t(\zeta)$ is in Region 1, Lemma 2.2 implies that $|h_r(\zeta)| < |\zeta - c\omega_l^*|^{r+2}$, where $c = 1$ or $\pm \frac{1}{3}$ is the center of the appropriate interval in Region 1. We define Region 2 to be the remaining three intervals of $[-1, 1)$, namely $[-\frac{2}{9}, \frac{2}{9}) \cup [\frac{4}{9}, \frac{8}{9}) \cup [-\frac{8}{9}, -\frac{4}{9})$. Note that the centers of these intervals, 0 and $\pm \frac{2}{3}$ are twice the centers of the intervals in Region 1. We have

$$(2.13) \quad t\left(\frac{\zeta}{2}\right) = \begin{cases} \frac{t(\zeta)}{2} \pmod{2} & \text{if } [|\zeta|] \in 2\mathbb{Z} \\ \frac{t(\zeta)}{2} + 1 \pmod{2} & \text{if } [|\zeta|] \in 2\mathbb{Z} + 1 \end{cases}$$

(where $[|\zeta|]$ denotes the greatest integer in $|\zeta|$.) Thus for a point with $t(\zeta)$ in Region 2, $\zeta \rightarrow \frac{\zeta}{2}$ either takes $t(\zeta)$ to a point in Region 1 that is half the distance from the center of its interval, or to a point in Region 2 that is half the distance from the center of its interval. Therefore, every time $t(\frac{\zeta}{2^k})$ lands in Region 1, we have $|h(\frac{\zeta}{2^k})| < \frac{1}{4^{j(r+2)}}$, where j is the number of times $t(\frac{\zeta}{2^k})$ landed in Region 2 since the last time it landed in Region 1. For a point with $t(\zeta)$ in Region 1, $\zeta \rightarrow \frac{\zeta}{2}$ takes $t(\zeta)$ farther away from the center, so we start over in measuring the degree of closeness. The pattern will continue until $|\frac{\zeta}{2^k}| < \frac{1}{8}$, at which point $t(\frac{\zeta}{2^k})$ can no longer get close to one of the centers. In the worst case, Regions 1 and 2 will alternate as $\frac{\zeta}{2^k}$ moves closer to the origin, in which case we will get a factor of $\frac{1}{4^{(r+2)}}$ for only every other k . The other factors are bounded by 1. Thus we have

$$(2.14) \quad \left| \hat{\phi}(\zeta) \right| < 2^{-(r+2)(\lfloor \log_2(|\zeta|) \rfloor + 2)} < |\zeta|^{-(r+2)},$$

which establishes that $\zeta^{r+1}\hat{\phi}_r(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$.

We use a similar technique to estimate the size of a partial derivative $D(\hat{\phi}_r)(\zeta)$ of order $s \leq r$. For any ζ , we let $k(\zeta) = \min\{k : |\frac{\zeta}{2^k}| < \frac{1}{8}\}$. By induction we see that $D \prod_{k=1}^{k(\zeta)-1} \frac{1}{2} h(\frac{\zeta}{2^k})$ has $(k(\zeta) - 1)^s < (\log_2(|\zeta|) + 3)^s$ terms, each of which has the sum of the orders of the derivatives represented equal to s . If we let M be the maximum value of all derivative of h of order less than or equal s in our neighborhood N of the central axis, then by an argument similar to that above, we have that

$$\begin{aligned} \left| D \prod_{k=1}^{k(\zeta)-1} \frac{1}{2} h\left(\frac{\zeta}{2^k}\right) \right| &< (\log_2(|\zeta|) + 3)^s \left(\frac{M}{2}\right)^s 2^{-2(\lceil \log_2(|\zeta|) \rceil + 2)} \\ &< \left(\frac{M(\log_2(|\zeta|) + 3)}{2}\right)^s \zeta^{-2} \\ &< \zeta^{-1} \end{aligned}$$

for ζ sufficiently large. Using (2.14), and the boundedness of $\prod_{k=k(\zeta)}^{\infty} \frac{1}{2} h(\frac{\zeta}{2^k})$ and its derivatives, this establishes $\zeta D(\hat{\phi}_r)(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$.

It remains to show that the multiplicity function for ψ is the same as the original multiplicity function m for the wavelet set wavelet. This requires two steps. The theorem in [5] that provides the basis for Theorem 2.1, only shows in general that the the constructed wavelet is *obtained from* the GMRA determined by ϕ , in the sense defined by Zalik [18]. This means only that the wavelet $\psi \in V_1$. We must show that under the current hypotheses, the constructed wavelet is also *associated with* the GMRA, so that V_j is the closed linear span of $\{\psi_{l,k}\}_{l < j}$. The second issue is that the multiplicity function for the GMRA determined by ϕ may in general be a degenerate form of the multiplicity function used to construct the filters. (An example is given in [5] where this is the case.) We will show that for the construction in this theorem, however, the multiplicity function is indeed the same as the multiplicity function for the wavelet set.

To see that ψ is associated with the GMRA determined by ϕ , we use the argument in [6, §4, Theorem 4]. For that argument to apply here, we must only show that $\text{Per}\phi \equiv \sum_{j \in \mathbb{Z}^2} |\hat{\phi}(\zeta + j)|^2$ is bounded. This follows immediately from $\zeta \hat{\phi}(\zeta) \rightarrow 0$, by breaking the sum up into dyadic intervals.

Finally we show that $m' \equiv$ the multiplicity function of the GMRA determined by ϕ is the same as m , the multiplicity function determined by the wavelet set (a.e.). By [2], m' is the characteristic function of the support of $\text{Per}\phi$. Since m and m' both take on only the values 0 and 1, it will suffice to show their supports are the same up to a set of measure 0. To see $\text{support}(m') \subset \text{support}(m)$, note that if $\hat{\phi}(\zeta + j) \neq 0$ for some $j \in \mathbb{Z}^2$, then $h(\frac{\zeta}{2} + \omega_l) \neq 0$ for some preimage ω_l of 0. but then $\zeta \in S_1 = \text{support}(m)$.

To get the opposite containment, suppose $\hat{\phi}(\zeta + j) = 0$ for all $j \in \mathbb{Z}^2$. We know $\text{support}(\hat{\phi}) = \cap_{j=1}^{\infty} \cup_{k \in \mathbb{Z}^2} \text{support}(h(\frac{\cdot}{2^j} + k))$, and by Lemma 2.2, $\text{support}(h) = P = S_1 \cap \{\zeta : 2\zeta \in S_1\}$ (up to a set of measure 0). Thus $\text{support}(\hat{\phi}) = \cap_{j=0}^{\infty} 2^j (\cup_{k \in \mathbb{Z}^2} (S_1 + k))$. This is exactly the set Δ defined in [9]. It is shown in [8] and also in [9] (in the context of the dimension function of a wavelet) that $m(\zeta) \leq \sum_{k \in \mathbb{Z}^2} \chi_{\Delta}(\zeta + k)$. Therefore $\hat{\phi}(\zeta + j) = 0$ for all $j \in \mathbb{Z}^2$ implies that $m(\zeta) = 0$.

□

REMARK 2.4. In general we will not have that $\text{Per}\phi$ or the corresponding $\text{Per}\psi$ are bounded away from 0 on their supports. Therefore by [11], the translates of neither ϕ nor ψ will form frames for their closed linear spans.

3. Examples

We now use the procedures outlined in the previous section to construct C^1 wavelets with C^1 transforms based on first the wedding cake wavelet set of [14] and then the diamond wavelet set of [16]. These wavelet sets are shown in Figure 1 below.

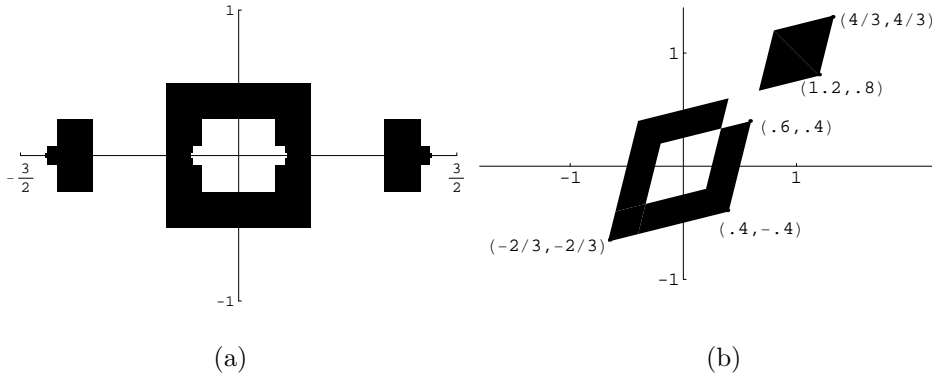


FIGURE 1. The wedding cake wavelet set and the diamond wavelet set.

EXAMPLE 3.1. Constructed by Dai, Larson and Speegle in [14], the wedding cake wavelet set (Figure 1(a)), was one of the earliest known examples of wavelet sets in \mathbb{R}^2 . The support of its multiplicity function, shown in Figure 2(a) below, has the x -axis as a central axis. The interior of S_1 consists of two star-shaped components (mod 1): one centered at the origin, and the other at the point $\omega_l^* = (\frac{1}{2}, 0)$. The set P is in this case $\frac{1}{2}S_1 \cup \frac{1}{2}S_1 + \omega_l^*$, represented by the black and gray regions respectively in Figure 2(b). The remaining requirements of star-simple, single-axis support are easily checked; in this case there are four transition points $(\pm\frac{1}{4}, \pm\frac{1}{16})$.

We build the filters h_1 and g_1 as described in Lemma 2.2. The low-pass filter h_1 will have value 2 at the origin and along the boundary between the gray and white regions in Figure 2(b); it will be 0 with first partials also equal to 0 at $\omega_l^* = (\frac{1}{2}, 0)$ and along the boundary between the black and white regions. Since h_1 is necessarily supported on P , h_1 will be discontinuous along the boundary between gray and white. However, $\zeta \rightarrow \frac{1}{2}\zeta$ takes these discontinuities to points on the boundary of the black where h_1 and its first partials are 0, canceling them out in the infinite product $\hat{\phi}$. The high-pass filter g_1 will have absolute value 2 between the dotted square and the black region in Figure 2(b) (since this is the region that is inside S_1 but outside P). Thus g_1 will be discontinuous along the dotted lines; this is cancelled by $\hat{\phi}$, which is 0 there (by h_1 being 0 at the boundary between black and white). The first quadrant of the filter h_1 is shown in Figure 3 below. The discontinuities of h_1 along the boundary of $\frac{1}{2}S_1 + \omega_l^*$ are apparent.

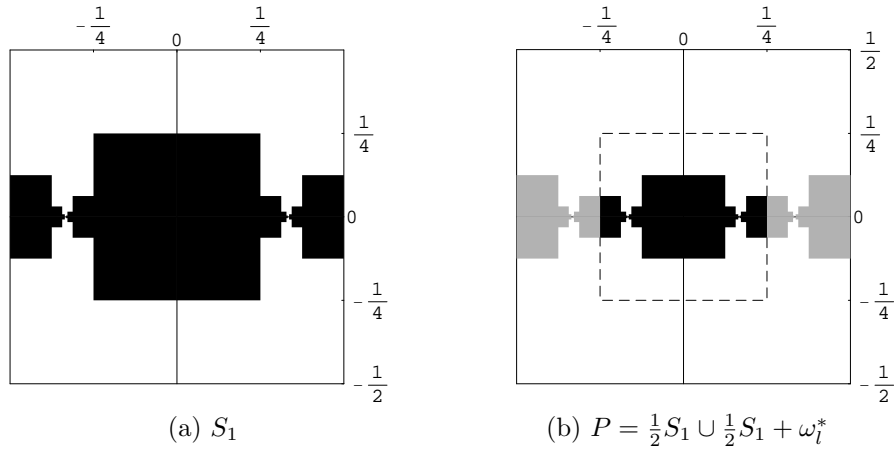


FIGURE 2. The support of the multiplicity function and of the low-pass filter h_1 for the wedding cake set construction.

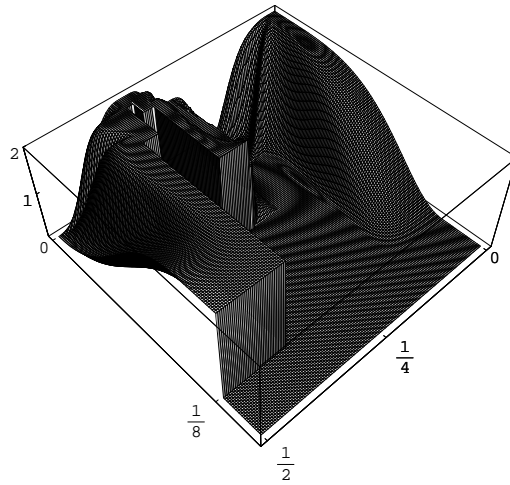


FIGURE 3. The first quadrant graph of the filter h_1 on the wedding cake wavelet set

As described in Section 2, the function $\hat{\phi}_1(\zeta) = \prod_{k=1}^{\infty} \frac{1}{2} h_1(\frac{\zeta}{2^k})$ smooths out the discontinuities of h_1 to create the C^1 function shown below in Figure 4.

EXAMPLE 3.2. We now apply the procedure of Section 2 to construct a C^1 wavelet with C^1 transform based on the diamond wavelet set of [16]. This wavelet set, shown in Figure 1(b), has the property that it is a finite union of convex polygons. Again the interior of the support of the multiplicity function has just two connected components, one centered at the origin and the other at $\omega_l^* = (\frac{1}{2}, \frac{1}{2})$. (See Figure 5(a).) Figure 5(b) shows $P = \frac{1}{2}S_1 \cup \frac{1}{2}S_1 + \omega_l^*$ in black and gray respectively. In this case, the central axis is the line $y = x$, and four transition points occur at

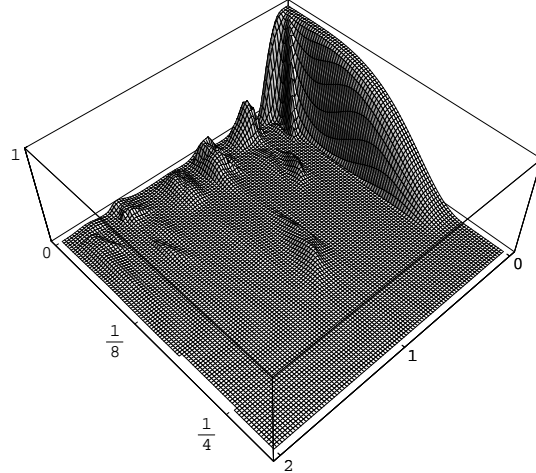


FIGURE 4. The first quadrant graph of $\hat{\phi}_1$ for the wedding cake set.

$\pm(.2, .3)$ and $\pm(.3, .2)$. Just as in Example 1, we will have h_1 discontinuous on the boundary between gray and white, and g_1 discontinuous along the dotted line.

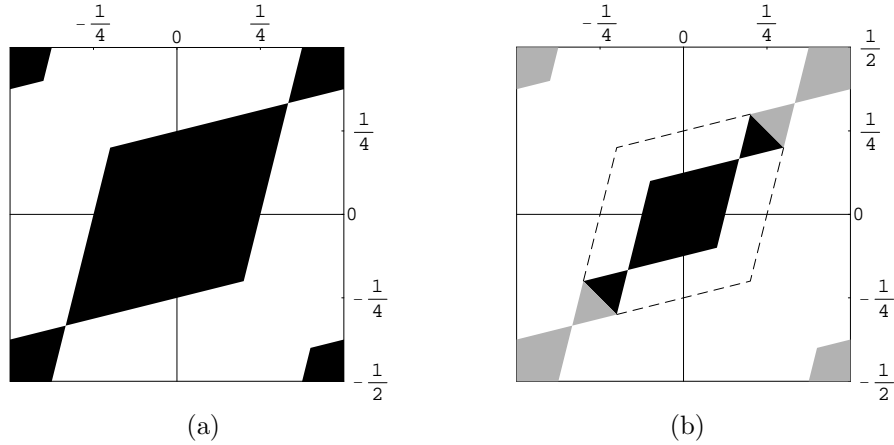


FIGURE 5. The support of the multiplicity function and of the low pass filter h_1 for the diamond wavelet set construction.

Figure 6 below is a graph of the filter h_1 for a C^1 with C^1 transform wavelet on the diamond wavelet set. To make the shapes clearer by including all of the outer diamonds, it is graphed on a slightly larger set than one period. The figure shows the discontinuities at the edges of the outer (gray) diamonds, and also the rapid change near the transition points.

As guaranteed by Theorem 2.3, we see in Figure 7 that the function $\hat{\phi}$ again smooths out the filter and is only supported along the central axis. (It is barely possible to pick out tiny bumps along the line $y = x$ beyond the central spire.)

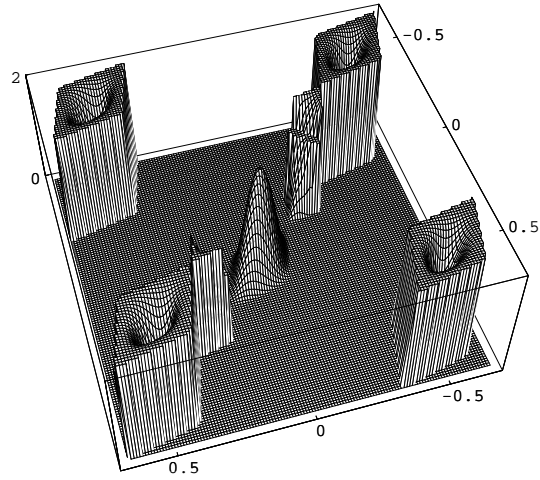
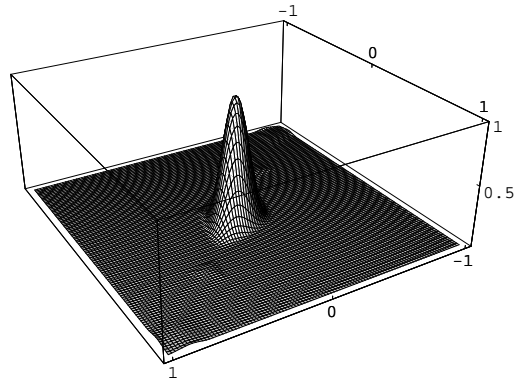


FIGURE 6. A filter on the diamond wavelet set

FIGURE 7. $\hat{\phi}$ for the diamond wavelet set

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