Simple wavelet sets for matrix dilations in $\mathbb{R}^2$

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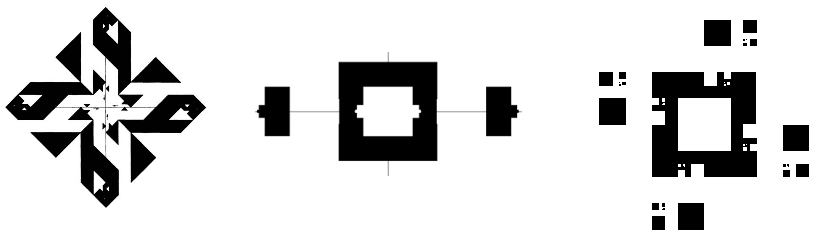
Abstract. A partial answer is given to the question of which expansive integer matrix dilations in $\mathbb{R}^2$ have wavelet sets that are finite unions of convex sets. New results are given supporting a conjecture that among matrices with determinant greater than 2, it is exactly matrices that have a power equal to a scalar that have such wavelet sets.

1. Introduction

A *wavelet set* relative to dilation by an expansive 2×2 integer matrix A is a set $W \subset \mathbb{R}^2$ whose characteristic function $\mathbf{1}_W$ is the Fourier transform of an orthonormal wavelet. That is, if $\hat{\psi} = \mathbf{1}_W$, then $\left\{\psi_{j,k} \equiv \sqrt{|\det A|^j}\psi(A^j \cdot -k),\ j \in \mathbb{Z}, k \in \mathbb{Z}^2\right\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$. In general, wavelets based on wavelet sets are not directly useful for applications because their discontinuous Fourier transforms imply that they cannot be well-localized. However, they have proved an essential source of examples and counterexamples used to develop the theory of wavelets. A famous wavelet set example due to Journé [**1**] first showed that not all wavelets have an associated structure called a multiresolution analysis (MRA). In 1997, Dai Larson and Speegle [**2**] used wavelet sets to show that single wavelets exist for an arbitrary expansive matrix in any dimension. In addition to their usefulness as examples or counterexamples, wavelet set wavelets have been used as building blocks for more well-localized wavelets (See e.g. [**3**], [**4**]), [**5**] , [**6**], [**7**].) Also, Lim, Packer and Taylor [**8**] were able to decompose wavelet representations as direct integrals of irreducibles over wavelet sets.

Early 2-dimensional examples of wavelet sets for dilation by 2 were produced by Zakharov [**9**], Soardi and Wieland [**10**], and Dai, Larson and Speegle [**11**]. In 1999, general construction techniques were introduced independently by Benedetto and Leon [**12**] and

Baggett, Medina and Merrill [**13**]. All of these construction techniques involve infinite iterative procedures, and all early examples show fingerprints of this procedure in the fractal-like structure of the wavelet sets. See Figure 1 below.



Soardi/Wieland Dai/Larson/Speegle Benedetto/Leon Baggett/Medina/Merrill

FIGURE 1. Early fractal-like wavelet sets for dilation by 2

Many early researchers believed that this complicated geometric structure was unavoidable. In [**14**], Benedetto and Sumetkijakan showed that a wavelet set for dilation by 2 in $\mathbb{R}^n$, $n \geq 2$, cannot be the union of n or fewer convex sets, and conjectured that it could not be the union of a finite number of convex sets. Soardi and Wieland in [**10**] made the weaker conjecture that a dilation by 2 wavelet set in dimension greater than 1 could not be the finite union of polygons.

However, in 2004, Gabardo and Yu [**15**] used self-affine tiles to produce a wavelet set for dilation by 2 that is a finite union of polygons. In 2008 [**16**], we used a different technique, based on generalized multiresolution analyses, to construct such wavelet sets for arbitrary real scalar dilations. The question has since been raised as to exactly which expansive (all eigenvalues great than 1 in absolute value) integer matrix dilations have wavelet sets that are finite unions of polygons or finite unions of convex sets. Simple wavelet sets for expansive integer matrices with determinant 2 had been known prior to 2004; using the idea of $\mathbb{Z}$-similarity, all determinant 2 matrices can be shown to have simple wavelet sets from the examples of representatives in Calogero [**17**], Gu and Han [**18**], and Bownik and Speegle [**6**]. In addition, two strikingly different simple wavelet sets for the matrix $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ were given in [**15**].

In this paper we extend these results to present a partial answer to the question of which expansive integer 2×2 matrices have simple wavelet sets. In Section 2, we show that the two characterizations of simple wavelet sets embodied in the conjectures of Benedetto/Sumetkijakan and Soardi/Wieland are actually equivalent. In Section 3, we produce simple wavelet sets for dilation by any matrix that has a nonzero power equal to a multiple of the identity, as long as it is $\mathbb{Z}$-similar to a matrix with singular values greater than $\sqrt{2}$. Section 4 contains our central negative result: that a matrix such that no power has real eigenvalues cannot have a simple wavelet set. Because of the importance of preserving slope in both the positive and negative results, we conjecture that the class of integer matrices with determinant greater than 2 having simple wavelet sets is exactly those with a nonzero power equal to a multiple of the identity.

2. Simple wavelet sets

While a wavelet set was originally defined as a set whose characteristic function is the Fourier transform of a wavelet, the following well known result gives an alternate geometric description.

THEOREM 2.1. *A measurable set $W \subset \mathbb{R}^2$ is a wavelet set for dilation by an expansive integer matrix A if and only if*

$$\sum_{k \in \mathbb{Z}^2} \chi_W(x+k) = 1 \qquad a.e.\ x \in \mathbb{R}^2 \tag{1}$$

$$\sum_{j \in \mathbb{Z}} \chi_W(A^{*j}x) = 1 \qquad a.e.\ x \in \mathbb{R}^2. \tag{2}$$

PROOF. See, e.g. [**5**]. □

In other words, W is a wavelet set if and only if it tiles the plane (up to sets of measure zero) under both translation and dilation by A^*. (The appearance of the transpose, A^*, is caused by the Fourier transform in the original definition of wavelet set.) Using this characterization, we now show that the two descriptions of simple wavelet sets that have previously appeared in the literature are equivalent.

Theorem 2.2. *Let A be an expansive 2×2 integer dilation matrix. A wavelet set for dilation by A that is a finite union of convex sets of positive measure must be a finite union of bounded convex polygons.*

Proof. Let W be a wavelet set of the form $W = \cup_{j=1}^{J} W_j$, where all of the W_j are convex sets of positive measure. First we show that the W_j must all be bounded. To see this, note that each W_j, as a set of positive measure, must contain a closed disk D_j. If some W_j is unbounded, and thus also contains a sequence of points $\{x_n\}$, such that $|x_n| \to \infty$, then the convex hull of $D_j \cup \{x_n\}$ will contain triangles of arbitrarily large area. This would force W to have infinite measure, which is impossible by the definition.

Now, suppose that one of the convex sets making up W is not a polygon. That is, suppose W_{j_1} has a piece c of its boundary that is not a finite union of line segments. Since the integer translates of W tile the plane and since W is bounded, there must be a finite number of translates of W with pieces of boundary that partition c . Let $\widetilde{W}_1, \widetilde{W}_2, \cdots \widetilde{W}_n$ be the nonzero translates of convex subsets of W that have c as part of their boundary, and let c_k be the subset of c that is the shared boundary with $\widetilde{W}_k$. Since c is not a finite union of line segments, one of these segments, c_{k_0} must not be a line. Thus the line between the endpoints of this c_{k_0} cannot lie in both W_{j_1} and $\widetilde{W}_{k_0}$. This contradicts the fact that both of these sets were assumed to be convex. $\square$

By definition, wavelet sets are only determined up to sets of measure zero. Thus, in writing a wavelet set as a finite union of convex sets or polygons, we may assume the component sets have positive measure and we may include part or all of their boundaries as we choose. Using Theorem 2.2, we make the following definition.

Definition 2.3. A wavelet set W for an expansive 2×2 integer matrix dilation A is called a *simple wavelet set* if W can be written as a finite union of convex sets, or equivalently, if W can be written as a finite union of bounded convex polygons.

Two matrices A and B are said to be $\mathbb{Z}$-*similar* if there is an integer matrix C with $\det C = \pm 1$ such that $A = CBC^{-1}$. Using Theorem 2.1, It is easy to see that if W is a wavelet set for B, then CW is a wavelet set for A. Thus the property of having a simple wavelet set is shared by all members of a $\mathbb{Z}$-similarity class. These similarity classes are described for matrices with small determinants in [**19**] and [**20**]. When B is a scalar matrix, we have $A = CBC^{-1} = A$, so the $\mathbb{Z}$-similarity class has only one element. In this case, considering $\mathbb{Z}$-similarity yields the result that if W is a simple wavelet set for a scalar matrix B, then CW will be also for any integer matrix C with $\det C = \pm 1$.

The results in this paper will be developed in terms of the geometric description of wavelet sets given in Theorem 2.1. However, in order to use the construction technique developed in [**13**] and [**16**], we will also need the following definition, which is motivated by the theory of multiresolution analysis.

Definition 2.4. A set $E \subset \mathbb{R}^2$ is called a *generalized scaling set* for dilation by A if $E = \cup_{j<0} A^{*j} W$ for some wavelet set W, or equivalently, if $E \subset A^* E$ and $A^* E \setminus E$ is a wavelet set.

The following result gives sufficient conditions for a set E to be a generalized scaling set. We will use these conditions to build wavelet sets in Section 3.

Lemma 2.5. *Suppose that A is an expansive integer matrix and that the measurable set $E \subset \mathbb{R}^2$ satisfies $E \subset AE$ and contains a neighborhood of the origin. Suppose further that $\mathbf{1_E}$ satisfies the consistency equation*

$$1 + \sum_{k \in \mathbb{Z}^2} \mathbf{1}_E(x + k) = \sum_{k \in \mathbb{Z}^2} \mathbf{1}_E((A^{-1}(x + k)) \quad a.e.. \tag{3}$$

Then E is a generalized scaling set for dilation by A.

Proof. See [**16**]. □

The function on $\mathbb{R}^2/\mathbb{Z}^2$ that occurs in the consistency equation of Lemma 2.5 is called the *multiplicity function*, $m(x) = \sum_{k \in \mathbb{Z}^2} \mathbf{1}_E(x + k)$.

3. Scalar-potent dilations

We now generalize the construction of simple wavelet sets for scalar dilations in [**16**] to scalar-potent dilations, i.e. matrices that have some integral power equal to a scalar.

Theorem 3.1. *Let A be an expansive integer matrix with all of its singular values greater than $\sqrt{2}$. If some integral power of A is multiple of the identity, then A has a simple wavelet set.*

Proof. Let $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = A^n$, where $\lambda > 0$. First, we build a generalized scaling set $E^\Lambda = \cup_{k=0}^{\infty} E_k^\Lambda$ for Λ as outlined in [**16**]. The first piece, E_0^Λ is a truncated diamond along the main diagonal, with center c at $(0,0)$ for even λ, and at $(\frac{-1}{2(\lambda^2-1)}, \frac{-1}{2(\lambda^2-1)})$ for odd λ. The cut off edges in the first and third quadrants are $\frac{1}{\lambda^2}$ times as long as the parallel centerline, which extends between $c \pm (\frac{\lambda}{2(\lambda^2+1)}, \frac{\lambda}{2(\lambda^2+1)})$. The second piece, $E_1^\Lambda = \frac{1}{\lambda}\left(E_0^\Lambda + ([\frac{\lambda}{2}], [\frac{\lambda}{2}])\right)$ (where$[\cdot]$ denotes the greatest integer function) is a smaller truncated diamond translated out into the first quadrant. The third piece splits into top and bottom halves $E_2^\Lambda = \frac{1}{\lambda}E_{1t}^\Lambda \cup \left(\frac{1}{\lambda}E_{1b}^\Lambda - (\frac{1}{\lambda}, \frac{1}{\lambda})\right)$ that match up with the two cutoff ends of E_0^Λ. The two halves of the remaining odd E_n's fill in the two truncated ends of the diamond E_1^Λ, while the halves of the even E_n's continue to fill in the truncated ends of E_0^Λ. Each of these E_n^Λ, $n > 0$, is translation congruent mod $\Lambda^{-1}\mathbb{Z}^2$ to E_{n-1}^Λ, and E_0^Λ is translation congruent modulo $\Lambda^{-1}\mathbb{Z}^2$ to $\Lambda^{-1}[-\frac{1}{2}, \frac{1}{2})^2$. These facts guarantee that the scaling set will satisfy the consistency equation (3).

The resulting generalized scaling set has the form

$$E^\Lambda = D_1 \cup D_2, \tag{4}$$

where D_1 and D_2 are diamonds that lie along the main diagonal through the origin. The larger has corners at $c \pm (\frac{\lambda}{2(\lambda^2-1)}, \frac{\lambda}{2(\lambda^2-1)})$ and $c \pm (\frac{\lambda}{2(1+\lambda^2)}, \frac{-\lambda}{2(1+\lambda^2)})$; the smaller is formed by taking $\frac{1}{\lambda}$ times the larger after first translating by $([\frac{\lambda}{2}], [\frac{\lambda}{2}])$. The simple wavelet set for Λ is then of the form

$$W^\Lambda = \lambda E^\Lambda \setminus E^\Lambda = (\lambda D_1 \setminus (D_1 \cup D_2)) \cup \lambda D_2. \tag{5}$$

This wavelet set has two pieces: a large diamond around the origin that is missing a smaller inset diamond and a notch, together with a second diamond congruent to the missing inset diamond translated out to $([\frac{\lambda}{2}], [\frac{\lambda}{2}])$. (See Figure 2 below and [**16**] for more details.)

We build the generalized scaling set for A, $E^A = \cup_{k=0}^{\infty} E_k^A$, from the pieces E_k^Λ as follows. Let $E_{2kn}^A = A^{*n-1}E_{2k}^\Lambda$ and $E_{(2k+1)n}^A = A^{*n-1}E_{2k+1}^\Lambda$ for $k \geq 0$. The intermediate pieces of E^A are defined in terms of these by $E_{2kn-j}^A = A^{*j}E_{2kn}^A$ for $-n < j < n$, so that A^{*-1} takes E_{2nk+l}^A to $E_{2nk+l+1}^A$ for $1 \leq l < 2n$. The resulting generalized scaling set E^A will have $2n$ pieces, each a (possibly distorted) diamond. In particular, using (4), we have

$$E^A = A^{*n-1}D_1 \cup \left(\cup_{0\leq l\leq 2n-2} A^{*l}D_2\right). \tag{6}$$

(See Figure 3 in Example 3.2 below.)

Note that the pieces E_k^A satisfy that $E_0^A = A^{*n-1}E_0^\Lambda$ is congruent modulo $A^{*-1}\mathbb{Z}^2$ to $A^{*-1}[-\frac{1}{2}, \frac{1}{2})^2$, and E_m^A is congruent modulo $A^{*-1}\mathbb{Z}^2$ to $A^{*-1}E_{m-1}^A$ for $m > 0$, since the pieces of E^Λ satisfy the corresponding congruences. Thus E^A satisfies the required consistency equation (3) as long as the pieces E_k^A are disjoint. To check this, it will suffice to show that $D_2 \cap A^*D_2 = \emptyset$. The distance between the centers of D_2 and A^*D_2 is on the order of $\frac{1}{\lambda}$, while their combined diameters are on the order of $\frac{1}{\lambda^2}$. Making λ larger will decrease the size of the second relative to the first, so we can make these pieces disjoint by using a larger power $A^{nj} = \Lambda^j$ if necessary in order to increase the size of λ.

Since D_1 clearly contains the origin, the set E^A will meet the criteria of Lemma 2.5 once we show $E^A \subset A^*E^A$. Using (6), we have $A^*E^A = \Lambda D_1 \cup \left(\cup_{1\leq l\leq 2n-1} A^{*l}D_2\right)$, so that it will suffice to show that $A^{*-1}(\Lambda D_1) \cup D_2 \subset \Lambda D_1$. Since we know from the scalar case that $D_2 \subset \Lambda D_1$, we need only show $A^{*-1}(\Lambda D_1) \subset \Lambda D_1$. Again by using higher powers $A^{nk} = \Lambda^k$ to form the diagonal matrix if necessary, we can make the diamond D_1 and thus ΛD_1 as close as we like to a square centered at the origin. Thus the longest vector in D_1 can be made arbitrarily close to $\sqrt{2}$ times the shortest vector, so that the fact that the smallest singular value of A is greater than $\sqrt{2}$ will imply the inclusion that we need.

Using the generalized scaling set E^A, the wavelet set W^A is defined by $W^A = A^*E^A \setminus E^A$. Similar to the wavelet set for Λ, it will consist of a notched (possibly distorted) diamond that is missing a smaller inset diamond that has been translated out. □

Example 3.2. Let $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, and $\Lambda = A^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. As outlined in Theorem 3.1, we start with the generalized scaling set for Λ. We have $E^\Lambda = D_1 \cup D_2$, where D_1 has corners at $(\frac{2}{15}, \frac{2}{15})$, $(\frac{2}{17}, \frac{-2}{17})$, $(\frac{-2}{15}, \frac{-2}{15})$, and $(\frac{-2}{17}, \frac{2}{17})$, and $D_2 = \frac{1}{4}D_1 + (\frac{1}{2}, \frac{1}{2})$. The wavelet set for Λ is then formed by taking $W^\Lambda = 4E^\Lambda \setminus E^\Lambda$. See Figure 2 below.

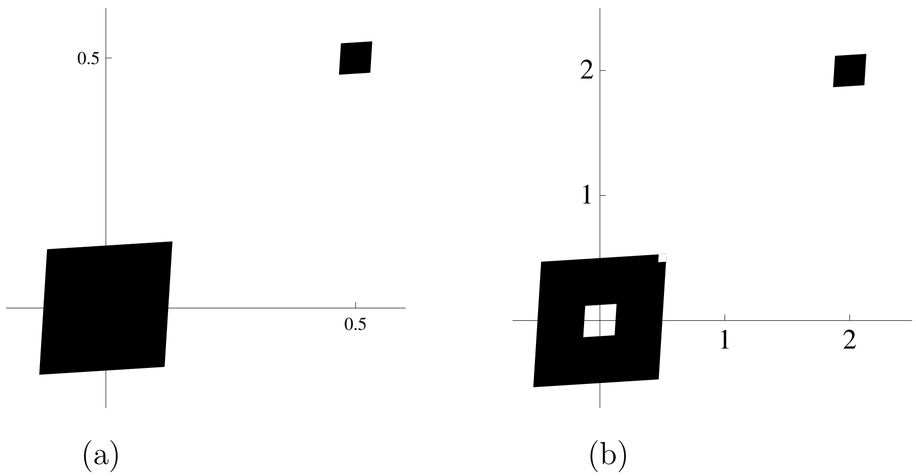


FIGURE 2. The generalized scaling set and wavelet set for Λ

The scaling set E^A for A is given by $A^*(D_1) \cup A^{*2}(D_2) \cup A^*(D_2) \cup D_2$; the wavelet set $W^A = A^*E^A \setminus E^A$. See Figure 3 below.

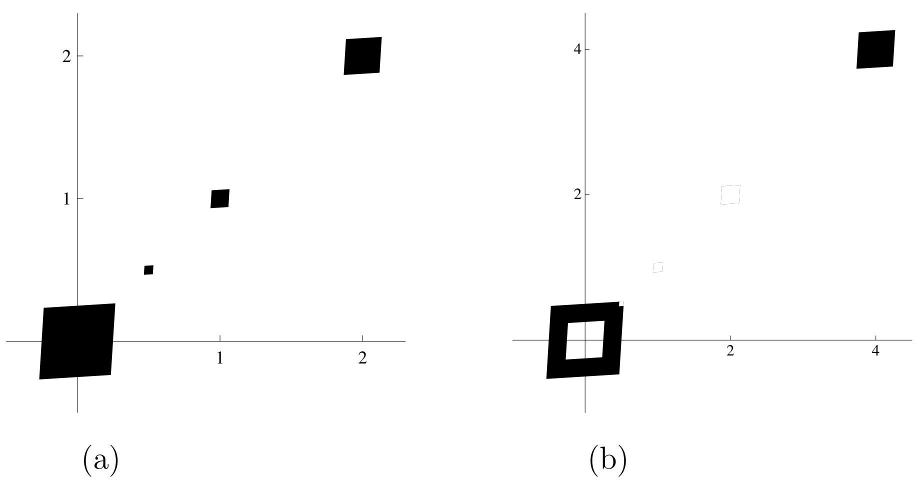
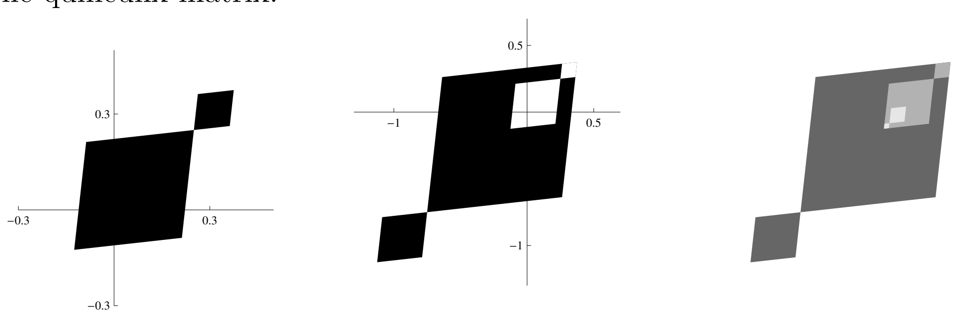


FIGURE 3. The generalized scaling set and wavelet set for A

REMARK 3.3. The proof of Theorem 3.1 makes use of the flexibility in the choice of the power on A used to get a scalar matrix. An extreme example of this flexibility is the fact that a scalar matrix itself can always be raised to powers to get other scalar matrices. In this way, we can get many different simple wavelet sets for dilation by a scalar. All have a similar shape, but ones using larger powers will have a smaller notch and the second diamond farther out in the first quadrant. For example, if we use the dilation by 4 wavelet set in Example 3.2 to build a wavelet set for dilation by 2, we get the same result as in Figure 3 (b) below, which is more square in shape and with its satellite farther out than the wavelet set for dilation by 2 given in [**16**]. If we use higher powers of 2, we will get wavelet sets for dilation by 2 that look more and more like the square $[\frac{-1}{2}, \frac{1}{2})^2 \setminus [\frac{-1}{4}, \frac{1}{4})^2$, with a barely discernible notch missing in the first quadrant, and a satellite congruent to the hole out farther and farther on the line $y = x$.

Matrices that have a power equal to a negative scalar can of course be handled by Theorem 3.1 by using twice that power to get a positive scalar. However, the number of pieces in the generalized scaling set and thus the size of the wavelet set can be cut in half by noting that $E^{-\Lambda} = -D_1 \cup D_2$ is a generalized scaling set for $-\Lambda$, where $E^{\Lambda} = D_1 \cup D_2$ is the wavelet set for Λ described by Theorem 3.1. This follows by mimicking the steps in the construction of E^{Λ} but replacing Λ by $-\Lambda$. Figure 4 below shows the generalized scaling set and wavelet set for $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ built in this way. The illustration of the dilation shows the interchange of the position of the two diamonds in each successive dilate. Example 3.5 uses this idea to build a wavelet set and generalized scaling set for twice the quincunx matrix.



E W Tiling by dilaton

FIGURE 4. Dilation by -3

REMARK 3.4. In Section 2, we mentioned that if W is a wavelet set for a scalar matrix Λ, then CW is as well for any integer matrix C with $\det C = \pm 1$. This can be used to produce other wavelet sets from the ones given above for both positive and negative scalar dilations. For example, by taking $C = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, we can move all the wavelet sets given above, which are centered on the diagonal, to wavelet sets centered on the x axis. When W is the wavelet set for dilation by 2 from [**16**] discussed above, CW is the wavelet set for dilation by 2 that appears in [**15**].

Example 3.5. Let $A^* = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$. Take $\Lambda = A^{*4} = \begin{pmatrix} -64 & 0 \\ 0 & -64 \end{pmatrix}$. The diagram on the left of Figure 5 below shows that the generalized scaling set for A has four diamonds: a large one containing the origin, a smaller one near $(2, -2)$, a third quite small near $(0, -1)$ and a fourth near $(-.25, -.25)$, barely visible in Figure 5. The dilate of this smallest just fits inside the corner of the first that is on the negative x-axis. (We built this scaling set using a slightly more off-center scaling set for Λ than the one outlined in Theorem 3.1 in order to keep the satellites closer in and thus make the picture clearer.) The wavelet set for A has two pieces as shown in the righthand diagram of Figure 5: an almost square with a diamond missing near the origin, and the missing diamond out near $(8, 0)$. Finally, the diagram in Figure6 shows a piece of the tiling by dilation. The large dark diamond is the larger of the two pieces of A^*W; the smaller piece of A^*W is a square congruent to the light colored square inside it, which is outside the range of the diagram to the upper right. This light colored square and the light colored diamond on the right are the two pieces of W that you see in the middle diagram. The midtone diamond inside the light square, together with the dot to the lower right together form $A^{*-1}W$.

The property of having a power equal to a scalar is easily seen to be shared by all matrices in the same $\mathbb{Z}$-similarity class. An examination of the list of the 11 similarity classes for expansive determinant 3 integer matrices given in [**20**] reveals that exactly 5 of these similarity classes have a power equal to a scalar matrix. However, all of the representatives in these similarity classes have a singular value less than $\sqrt{2}$, so that Theorem 3.1 cannot be used. For example, Figure below shows the failure of Theorem

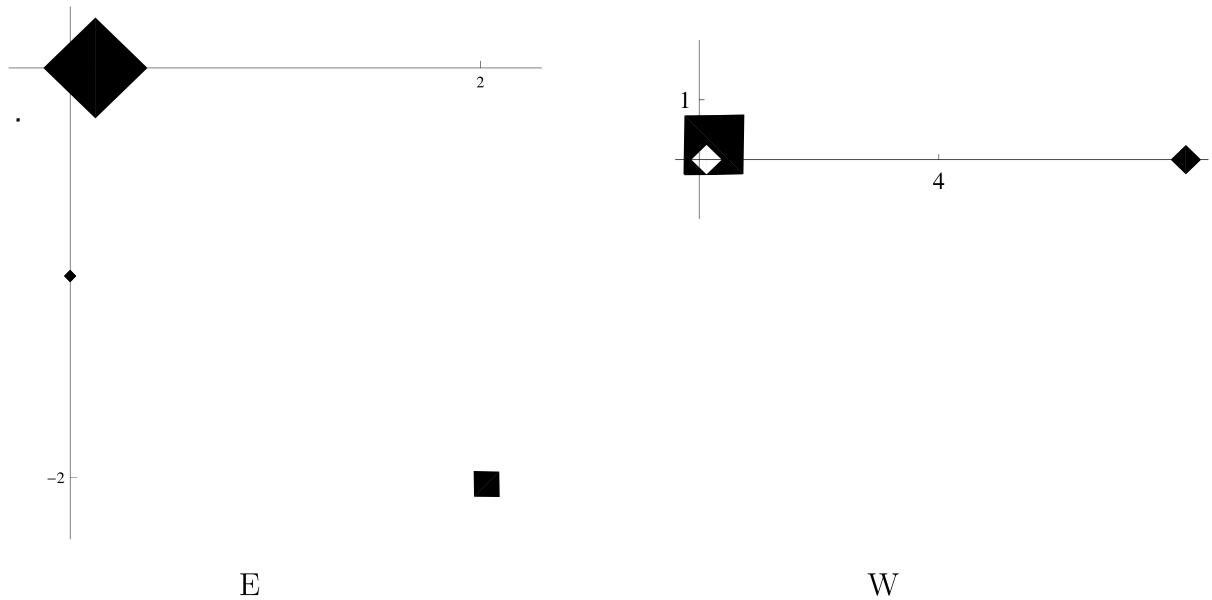


FIGURE 5. Dilation by 2Quincunx

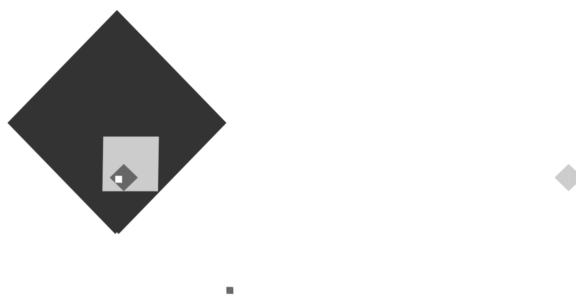


FIGURE 6. A piece of the dilation tiling for 2Quincunx

3.1 for the matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$, with singular values of 2.3 and 1.3. We have $A^2 = \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Here the potential scaling set $E = A^*(D_1) \cup A^{*2}(D_2) \cup A^*(D_2) \cup D_2$ does not satisfy $E \subset A^*E$ because $A^*D_1 \not\subset A^{*2}D_1$. This does not mean that this A does not have a simple wavelet set, or even necessarily that the construction technique will not work on a different member of the $\mathbb{Z}$-similarity class. The condition that the matrix have all singular

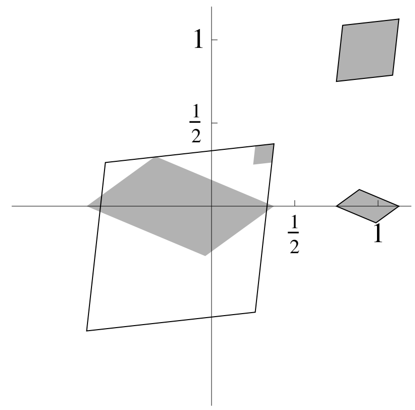


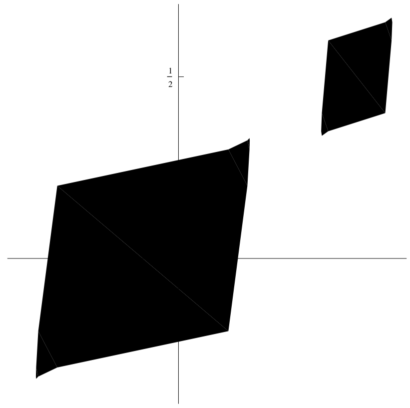
FIGURE 7. E (in gray) not contained in AE (in outline)

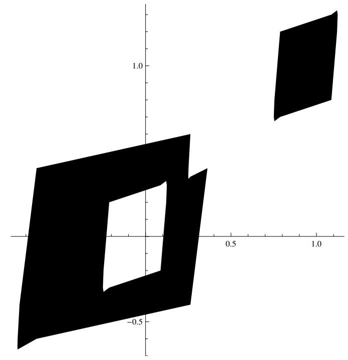
values greater than $\sqrt{2}$ is sufficient, but not necessary, since the longest vector in E may not always line up with the shortest in A^*E.

4. Dilations that are not scalar-potent

The construction technique used in [**16**] and extended in the previous section, avoids the fractal-like shape that naturally occurs from an infinite iterated process by forcing later pieces in that process to join smoothly with earlier ones. This is done by choosing parameters (the center of the first piece and the length of its truncated side) so that a later piece matches up in both position and length of the joining side. However, to have the end result be a finite union of convex polygons, the slopes of the adjacent sides also must match up. This occurs when the dilation matrix is a scalar, or for an appropriately chosen piece, when it has a power equal to a scalar. If we use the same technique for a non-scalar-potent dilation, we get a scaling set, and thus a wavelet set, that has a similar shape, but is not a finite union of convex sets because of curling caused by changing slopes of the joined pieces. This is illustrated in Example 4.1 below.

EXAMPLE 4.1. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Then A does not preserve any slopes except 0 and ∞. Thus, if we use our construction procedure to form a generalized scaling set and then wavelet set, the slopes of the sides of the pieces do not match up, giving a curl to the ends of the diamonds. This is illustrated in Figure 8 below.





(a) (b)

FIGURE 8. A generalized scaling set and wavelet set that are not finite unions of convex sets.

Next, we show that this difficulty is not entirely an artifact of this particular construction technique. That is, for matrices that have no power preserving any slope, there cannot be a simple wavelet set.

LEMMA 4.2. *Let A be an expansive integer dilation matrix such no nonzero power of A has real eigenvalues. Suppose W is a wavelet set for A that can be written as a finite union of bounded convex polygons, $W = \cup_{j=1}^{n} W_j$. Then the scaling set E for W and the regions of constant value of the multiplicity function m can each also be written as a finite union of bounded convex polygons.*

PROOF. First we claim that there exists a neighborhood N of 0 such that $W \cap N = \emptyset$. Suppose not. Then 0 must be in the closure of one of the bounded convex polygons, say W_{j_0}. Since 0 is fixed by A^*, the sets $(A^*)^k W_{j_0}$, $k \in \mathbb{Z}$ must all contain 0. Since these sets are polygons, each set $(A^*)^k W_{j_0}$ contains at least a wedge $S_k = \{(x, y) : m_k x < y < m_{k+1} x,\ 0 \le x < c_k\}$. Since the dilates are disjoint up to sets of measure 0, these open wedges must be pairwise disjoint. However, by hypothesis, A must have two complex eigenvalues that are complex conjugates, $re^{i\theta}$ and $re^{-i\theta}$, with $|r| > 1$. Let $\Lambda = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. Then $A^n = r^n T \Lambda^n T^{-1}$ for some nonsingular complex matrix T. Since no power of A has real eigenvalues, we must have θ irrationally related to 2π. This means that we can find an integer n_0 such that $\|T\Lambda^{n_0}T^{-1} - I\| < |\frac{m_1c_0 - m_0c_0}{2}|$, thus contradicting the fact that the wedges S_{n_0} and S_0 are disjoint.

Thus, we have shown that there exists $m, M \in \mathbb{R}$ such that $x \in W$ implies $m < \|x\| < M$. Next we use this to prove that the scaling set E for W must contain a neighborhood of 0. There exists an integer J such that $\|A^{-j}\| < \frac{m}{M}$ for $j > J$. We then have $\|A^j x\| > m$ for $j > J$ and $x \in W$. The scaling set E can be written $E = \left(\cup_{j<J+1}(A^T)^j W\right) \setminus \left(\cup_{j=0}^{J+1}(A^T)^j W\right)$. The first set contains the neighborhood $N = \{x : \ \|x\| < m\}$, and the second excludes $\cap_{j=0}^{J+1}(A^*)^j N$.

Now we have $E = \cup_{n=1}^{\infty} A^{-n} W$, and since E includes a neighborhood of 0, only finitely many of these dilates of W contribute their boundary lines to the boundary of E. Thus E is also a finite union of bounded polygons. Finally, since E must also be bounded, we have that the region of constant value of the multiplicity function, $m(x) = \sum_{k\in\mathbb{Z}^2} \mathbf{1}_E(x+k)$, will also each be a finite union of convex polygons. $\square$

THEOREM 4.3. *Let A be an expansive integer matrix with $|\det A| > 2$. If $\lambda^n \notin \mathbb{R}$ for all $\lambda \in$ spectrum(A) and all $n \in \mathbb{Z}$, then A does not have a wavelet set that is a finite union of convex polygons.*

PROOF. Suppose A does have a wavelet set that is the finite union of convex polygons. By Lemma 4.2, the multiplicity function m associated with this wavelet set must have its regions of constant value each a finite union of convex polygons. Since $|\det A| > 2$, m is not constant, and so the boundaries of its regions of constant value must consist of a finite nonempty collection of line segments. A boundary line segment l is characterized by the condition that for each point $x \in l$, every neighborhood of x must contain points where m takes on two different values. Let l^0 be one of these boundary line segments.

We will make use of the consistency equation for multiplicity functions, from which (3) is derived (see [**13**]):

$$m(x) + 1 = \sum_{A^* z = x} m(z) \ a.e. \ x \in \mathbb{R}^2/\mathbb{Z}^2. \tag{7}$$

Let $x \in l^0$. By Equation (7), at least one of the $d = |\det A|$ preimages of x under A^* in $\mathbb{R}^2/\mathbb{Z}^2$ must have every neighborhood containing points where m takes on two different values. Now consider the d preimages of l^0 under A^*. These preimages are the line segments of the form $A^{*-1}(l^0 + z_k)$, where $\{z_1, z_2, \cdots z_d\}$ are a set of coset representatives for $\mathbb{Z}^2/A^*\mathbb{Z}^2$. One of these preimages must have a subsegment l^1 such that every point of l^1 has every neighborhood containing points where m takes on two different values. That is, l^1 must also be a boundary (sub)segment of regions of different values of m.

Repeat this argument to produce a sequence of boundary subsegments, $\{l^k\}$, such that $A^{*k}l^k$ lies along l^0. Let v be a unit vector in the direction of l^0. Since there are only a finite number of boundary lines in the support of m, v must be an eigenvector with real eigenvalue for some power $(A^*)^j$ of A^*. □

Remark 4.4. The fact that determinant 2 matrices are a special class is caused by the multiplicity function being identically 1 in that case. As was noted in the introduction, it is known that all determinant 2 matrices in dimension 2 have simple wavelet sets.

Example 4.5. If an expansive A with det > 2 has complex eigenvalues whose arguments are of the form $\tan^{-1}(\frac{p}{q})$, then A satisfies the hypothesis of the theorem. For example, $A = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}$ has eigenvalues $4 \pm 3i$ and thus has no wavelet set that is a finite union of polygons. Since the singular values of this matrix are both 5, this matrix does have a curly wavelet set similar to the one shown in Figure 8. Because $\det A = 25$ is relatively large, the curl would be hard to see.

Remark 4.6. In this paper, we have focussed on integer dilations. However, in [**16**] the construction was carried out for rational scalar dilations as well. Theorem 4.3 above also does not require A to have integer entries, so that, for example, we know that $A = \begin{pmatrix} 2 & \frac{-7}{12} \\ \frac{7}{12} & 2 \end{pmatrix}$ also has no simple wavelet set.

Acknowledgments

The author wishes to thank the referee for several suggestions that clarified the presentation.

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