KEY

Linear Algebra–Block 5, 2013

Midterm Exam

Take the midterm in Tutt Science Center. You may not use books, notes (except for the one 3"x5" card you prepared for yourself in advance), calculators or computers. Show all your work. Do not discuss the exam with anyone except me. Sign the Honor Code when you are finished, and bring the exam, along with your 3x5 card to my office (TSC 204). The exam is due by 4PM.

1. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

a. Find $\det(A)$

Along second row $= -2(4-3) - 2(0) = \boxed{-2}$

b. Find the inverse of A.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 4 & 0 & -1 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

c. If X is a 3x3 matrix such that $XA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, find AX.

$$AX = AXAA^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & \frac{1}{2} & -1 \\ 1 & -\frac{1}{2} & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 0 & 0 \\ 8 & 0 & 0 \\ 21 & 0 & 0 \end{bmatrix}}$$

2. Tell whether each of the following statements is always true. If true, give a proof; if false give a specific counterexample.

a. If A and B are 3x3 matrices with $AB = 0$ then either A or B is singular (non-invertible).

True $\det(AB) = 0$

so $(\det A)(\det B) = 0$

so either $\det A$ or $\det B$ must be 0

b. If A is any $3x3$ matrix and E is an $3x3$ elementary matrix then $\det(EA) = \det A$.

False example: $E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. If $w \in \mathbb{R}^3$ is a linear combination of u and v, then v is a linear combination of u and w.

False $w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

3. For each of the following conditions, tell whether it is possible to find an $3x3$ matrix A to make the condition hold. If so, find an example of a matrix A where it holds. If not, tell why it is impossible.

a. For every $\vec{b}$, $A\vec{x} = b$ has an infinite number of solutions.

Impossible: Consistent for every $b \Rightarrow$ pivot in every row of A

$\Rightarrow$ 3 pivots

$\Rightarrow$ pivot in every column of $A \Rightarrow$ no free variables

b. For some vectors $\vec{b}$, $A\vec{x} = b$ has an infinite number of solutions, while for other vectors $\vec{b}$, it has no solutions.

Possible Example: $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \end{bmatrix}$ only consistent if $b_2 = 2b_1$ and $b_3 = 3b_1$

has 2 free variables

c. For some some vectors $\vec{b}$, $A\vec{x} = b$ has an infinite number of solutions, while for others it has a unique solution.

Impossible: If unique solution for any b then only trivial solution to homogeneous

so unique solution for every b

4. Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ that first rotates points by an angle $\frac{\pi}{2}$ around the y axis (ccw when looking from the positive y axis), and then projects onto the xy plane.

a. Find the matrix of T.

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \to \begin{bmatrix}0\\0\\-1\end{bmatrix} \to \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} \to \begin{bmatrix}1\\0\\0\end{bmatrix} \to \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \text{So } \begin{bmatrix}0&0&1\\0&1&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix} \to \begin{bmatrix}0\\1\\0\end{bmatrix} \to \begin{bmatrix}0\\1\\0\end{bmatrix}$$

b. Is T one to one? Is T onto? Give reasons for your answers.

Not 1-1 - not a pivot in every column

Not onto - not a pivot in every row

5. Let $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $v_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $v_3 = \begin{bmatrix}3\\4\\7\end{bmatrix}$

a. Do these vectors span $\mathbb{R}^3$?

$$\begin{bmatrix}1&1&3\\2&0&4\\3&1&7\end{bmatrix} \to \begin{bmatrix}1&1&3\\0&-2&-2\\0&-2&-2\end{bmatrix} \to \begin{bmatrix}1&1&3\\0&1&1\\0&0&0\end{bmatrix}$$

no - not a pivot in every row

b. Are these vectors independent?

no - not a pivot in every column

c. If possible, write $\begin{bmatrix}0\\1\\1\end{bmatrix}$ as a linear combination of v_1 and v_2.

$$\begin{bmatrix}1&1&0\\2&0&1\\3&1&1\end{bmatrix} \to \begin{bmatrix}1&1&0\\0&-2&1\\0&-2&1\end{bmatrix}$$

$x = \frac{1}{2}$

$y = -\frac{1}{2}$

$$\text{So } \begin{bmatrix}0\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}$$