KEY

Linear Algebra–Block 5, 2013

Final Exam

Take the exam in Tutt Science Center. You may not use books, notes (except for the one 8-1/2"x11" sheet you prepared for yourself in advance), calculators or computers. Show all your work. Do not discuss the exam with anyone except me. Sign the Honor Code when you are finished, and bring the exam, along with your page of notes to my office (TSC 204). The exam is due by 1PM.

16pts 1. $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$. $\to \begin{bmatrix} 1 & 3 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

z Free
$y+z=0 \quad y=-z$
$x+3y+2z=0$
$x=z$

(6pts)a. Find bases for Col$A$, Row$A$, and Nul$A$.

Col A: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ Row A: $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ -4 \\ -4 \end{bmatrix}$ Nul A: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Which two of the subspaces in part a are orthogonal? Use your bases to prove they are.

(2pts)b. Use your bases to show that an appropriate two of the subspaces in part a are orthogonal

Nul A and Row A $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -4 \\ -4 \end{bmatrix} = 0$

(2pt) c. What is rank(A)? 2

(6pts)c. One of the vectors $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ is in Col$A$ and one is not. For the vector that is in Col$A$, find the general solution to $A\vec{x} = \vec{b}$, and express it in terms of Nul(A). For the vector that is not in Col$A$, find the general least squares solution to $A\vec{x} = \vec{b}$.

$\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \in$ col(A) $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & -1 & -2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -4 & -4 & -4 \\ 0 & -2 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

z Free
$y+z=1$ so $y=1-z$
$x+3(1-z)+2z=4$
$x=1+z$

gen soln $\boxed{\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \text{Nul}(A)}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ by least squares $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 3 & 3 & 0 \\ 3 & 11 & 8 \\ 0 & 8 & 8 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 3 & 3 & 0 & 1 \\ 3 & 11 & 8 & 3 \\ 0 & 8 & 8 & 2 \end{bmatrix} \to \begin{bmatrix} 3 & 3 & 0 & 1 \\ 0 & 8 & 8 & 2 \\ 0 & 8 & 8 & 2 \end{bmatrix}$

z is free
$y = \frac{1}{4} - z$
$3x = 1 - 3(\frac{1}{4}) + 3z$
$x = \frac{1}{3} - \frac{1}{4} + z$

$\boxed{\hat{x} = \begin{bmatrix} 1/12 \\ 1/4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$

2. (3pts each) Tell whether each of the following statements is always true. If true, give a proof; if false give a specific counterexample.

a. There can be no basis for M_{2x2} which includes the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 2 & 0 \end{bmatrix}$, and $\begin{bmatrix} 4 & -1 \\ 7 & 3 \end{bmatrix}$. Write the three matrices' coordinate vectors in terms of the usual basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

we get $\begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -1 \\ 2 & 2 & 7 \\ 1 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix}$ so coordinative vectors not independent so three matrices not independent

so TRUE

b. If two nxn matrices A and B are the same except that the first and second rows are switched, then A and B have the same eigenvalues.

False Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ eigenvalue 1, multiplicity 2

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $-\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$

eigenvalues $\lambda = \frac{1 \pm \sqrt{1+4}}{2}$

c. If a matrix A has $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ in its nullspace, then it cannot have $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ in its column space.

False Ex $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

d. If A is an 4×6 matrix, then there must be a nontrivial solution to the homogeneous equation $A\vec{x} = 0$.

$n = 6$ rank ≤ 4 so $\dim(\mathrm{nul}\, A) \geq 6 - 4 = 2$

so TRUE

3. Suppose that each year, $\frac{1}{10}$ of the US residents living outside of Colorado move in to the state, while $\frac{9}{10}$ continue to live outside the state. Also each year, $\frac{2}{10}$ of the people inside Colorado move out, while $\frac{8}{10}$ stay in the state. (We are ignoring international migration, and using somewhat unrealistic numbers to make the arithmetic easier).

(3 pts) a. Find the stochastic matrix A that describes the population movement.

Let $x_k = \begin{bmatrix} in_k \\ out_k \end{bmatrix}$ $\quad A = \begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix}$ (or if you let $x_k = \begin{bmatrix} out_k \\ in_k \end{bmatrix}$, then $A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$)

(3 pts)b. Use matrix diagonalization to find the general solution $\vec{x}_k$ to the equation $\vec{x}_{k+1} = A\vec{x}_k$ in terms of eigenvalues and eigenvectors.

det $\begin{bmatrix} .8-\lambda & .1 \\ .2 & .9-\lambda \end{bmatrix} = \lambda^2 - 1.7\lambda + .7 = 0$ $\quad \lambda = 1, \lambda = .7$

For $\lambda = 1$ evector $-.2x + .1y = 0$ so $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda = .7$ evector $.1x + .1y = 0$ so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$x_k = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{where } \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [x_0]_{\{[1\ 2]^T, [1\ -1]^T\}}$$

(3 pts) d. If the same matrix continues to describe population movement over many years, what percentage of the population would eventually live in Colorado?

as $k \to \infty$, $c_2(.7)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to 0$

so $x_k \to c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so $1/3$ will live in Colorado

(sounds pretty crowded!)

4. (3 pts)a. Find the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the vector $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, that is find $\text{proj}_{\vec{u}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$$\frac{1-2+3}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

(3pts) b. Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by $T(\vec{y}) = \text{proj}_{\vec{u}}(\vec{y})$, where $\vec{u}$ is as in part a. Prove that T is a linear transformation.

$$T(\vec{y}) = \frac{y \cdot u}{u \cdot u}\vec{u}$$

$$T(\vec{y}_1 + \vec{y}_2) = \frac{(\vec{y}_1+\vec{y}_2)\cdot u}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{\vec{y}_1 \cdot u}{u \cdot \vec{u}}\vec{u} + \frac{\vec{y}_2 \cdot \vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = T(\vec{y}_1) + T(\vec{y}_2)$$

$$T(cy) = \frac{(cy)\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{c\,(y\cdot u)}{\vec{u}\cdot\vec{u}}\vec{u} = c\,T(\vec{y})$$

(3pts)c. Find the matrix of T (defined in part b.)

$$T(e_1) = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad T(e_2) = \frac{-1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad T(e_3) = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$

(4pts)d. Is T one to one? Is T onto? Give reasons for your answers.

Not 1-1 $\quad [T] \to \begin{bmatrix} 1/3 & -1/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Not a pivot in each colum

Not onto $\quad$ Not a pivot in every row