THE CONSTRUCTION OF WAVELETS FROM GENERALIZED CONJUGATE MIRROR FILTERS IN $L^2(\mathbb{R}^n)$

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Abstract. The classical constructions of wavelets and scaling functions from conjugate mirror filters are extended to settings that lack multi-resolution analyses. Using analogues of the classical low-pass/high-pass filter conditions, generalized mirror filters are defined in the context of a generalized notion of multi-resolution analysis. Scaling functions are constructed from these filters using an infinite matrix product. From these scaling functions, non-MRA wavelets are built, including one whose Fourier transform is infinitely differentiable on an arbitrarily large interval.

1. INTRODUCTION

Our primary aim in this paper is to extend the work begun in [Cou] generalizing the famous techniques of Mallat, Meyer, and Daubechies for constructing wavelets and scaling functions from conjugate mirror filters. (See [Mal1], [Mey], and [Dau].) While their constructions always give so-called MRA wavelets, i.e., wavelets that are associated to a multiresolution analysis and a scaling function, our generalizations of these constructions produce non-MRA wavelets as well. The techniques of theirs that we have in mind can briefly be summarized as follows: Begin with a periodic function h that is a conjugate mirror filter, i.e., satisfies a certain "mirror" equation. Suppose that this filter h is chosen in such a way that the infinite product $\prod_{j=1}^{\infty} h(2^{-j}(\xi))$ converges to a nonzero function $\widehat{\phi} \in L^2(\mathbb{R})$, and set ϕ equal to the inverse Fourier transform of $\widehat{\phi}$. Next, construct from the function h a function g, that is a sort of "complementary mirror filter" to h, i.e., satisfies a certain kind of pointwise orthogonality condition relative to h. Finally, define a function ψ to be the dilate of the inverse Fourier transform of $g\widehat{\phi}$. Then, the function ϕ is a scaling function for a multiresolution analysis $\{V_j\}$, and the function ψ is an associated orthonormal wavelet. If the filter h is carefully chosen, say to be a trigonometric polynomial and to have a prescribed number of vanishing moments, then the resulting wavelet and scaling function can be shown to be smooth and have compact support.

Obviously, the ingenious part of this classical construction lies in cleverly choosing the initial function h so that the infinite product converges to a function with desirable properties. The second step, building the complementary function g, is a considerably easier problem. It amounts to constructing a unitary matrix-valued

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function whose first column is the vector $(h(\omega), h(\omega+\pi))$. In any case, the entire construction procedure suggests that these conjugate mirror filters play a fundamental role in the theory of MRA wavelets. Moreover, a relationship between filters and MRA's also holds in the reverse direction, since every MRA $\{V_j\}$ and scaling function ϕ determine a unique conjugate mirror filter.

These classical constructions were specifically for dilation by 2 in $L^2(\mathbb{R})$, and more important from our perspective, the resulting wavelet ψ was always an MRA wavelet. We wish to extend these constructions to dilations determined by arbitrary expansive integral matrices in $L^2(\mathbb{R}^n)$, and to do it in such a way that we obtain construction procedures for very general wavelets or multiwavelets, including non-MRA wavelets. We believe that this kind of general construction is even of interest for the classical dilation by 2 case in $L^2(\mathbb{R})$.

Many non-MRA wavelets have been constructed since the famous example of Journé appeared in [Mal2]. (See e.g. [DLS], as well as [BMM] and [BL].) However, most of the known examples come from wavelet sets, i.e., sets E for which the inverse Fourier transform of the indicator function χ_E is a wavelet. These examples, having Fourier transforms that are not continuous, necessarily fail to vanish rapidly at infinity, although they are for the most part real analytic. Indeed, it is known that any wavelet having both a minimal amount of smoothness and decay properties is necessarily an MRA wavelet. (See [HW].) Our goal in this paper is not simply to construct new wavelets, but rather to find more subtle examples between well-behaved MRA examples and the wavelet set examples. In particular, we are able to construct a non-MRA wavelet in $L^2(\mathbb{R})$ whose Fourier transform is C^∞ on an arbitrarily large interval (a, b).

To effect a generalization of the classical constructions, we first introduce the notion of a generalized conjugate mirror filter, which is a matrix $\{h_{i,j}\}$ of periodic functions satisfying a generalized form of the classical mirror equation. We are led to this notion of a generalized filter from the study of generalized multiresolution analyses (GMRAs). A GMRA is a sequence $\{V_j\}$ of closed subspaces of $L^2(\mathbb{R}^n)$, very like an MRA, except that instead of having the property that the subspace V_0 contains a scaling function, it is assumed only that V_0 is invariant under all translations by lattice points. Every multiwavelet, MRA or not, determines in a natural way one of these GMRAs. While the classical techniques produce an MRA from the filter, our generalization will produce a GMRA from the generalized filter.

If $\{V_j\}$ is a GMRA, then the resulting unitary representation of the lattice group $\mathbb{Z}^n$ acting on V_0 determines what is to us the fundamental object of interest for GMRAs, a unique "multiplicity function" m mapping the cube $[-\pi, \pi)^n$ into the set $\{0, 1, 2, \ldots, \infty\}$. This multiplicity function m is identically 1 if and only if the GMRA is actually an MRA. The basic development of this approach to generalized multiresolution analyses was first presented in [BMM], and further investigations can be found among other places in [Bag], [BM], [Cou], and [Web]. In the classical case of dilation by 2, the multiplicity function coincides with the "dimension function" of Auscher (See [Aus] [HW], and [Web]), and it is very likely that the general multiplicity function coincides with an appropriately generalized notion of the dimension function. (See [Cal] and [BRS].) Even so, the fact that these two functions are defined very differently makes the information that each provides useful in quite different ways.

In the next section, we spell out in some detail the general properties of GMRAs, generalized conjugate mirror filters and generalized scaling functions. In particular, we use the multiplicity function m associated to a GMRA $\{V_j\}$ to show the existence of a set $\{\phi_i\}$ of generalized scaling functions in V_0. Their translates are not orthonormal, but they do constitute a normalized tight frame for V_0. From these ϕ_i's, and the fact that V_0 is contained in the dilate of V_0, we show the existence of a matrix of periodic functions $\{h_{i,j}\}$ on $\mathbb{R}^n$ that plays the role of a generalized conjugate mirror filter. Given such a collection of $h_{i,j}$'s, we show how to construct a generalized complementary conjugate mirror filter, another matrix of functions $\{g_{k,j}\}$, and from them a frame multiwavelet associated to the given GMRA. In this generalized situation, the construction of these complementary functions is no longer as simple and direct as it was in the classical case. It is not just a matter of constructing a matrix-valued function from a given row or rows, for the "dimension" of the "matrix" changes with the point ω. The first instance of solving this problem of constructing the $g_{k,j}$'s was given in [Cou], where it was applied specifically to the GMRA (non-MRA) determined by the Journé wavelet.

The discussion in Section 2 assumes that we are given a GMRA to begin with. Section 3, on the other hand, contains our generalizations of the classical techniques for constructing scaling functions and thus the GMRAs themselves from filters. That is, we show that given a multiplicity function m, i.e., a function that satisfies known necessary conditions established in [BM] and [BRS], we can always construct a generalized conjugate mirror filter $\{h_{i,j}\}$ relative to m. Further, we can build this GCMF $\{h_{i,j}\}$ with properties that guarantee that a certain infinite "matrix" product converges. If we then define functions $\{\widehat{\phi_i}\}$ as the first column of this infinite product matrix, these ϕ_i's form generalized scaling functions for a GMRA whose associated multiplicity function is the given m. We can then use the results of Section 2 to define functions $\{\psi_k\}$, in terms of the generalized scaling functions $\{\phi_i\}$ and the complementary mirror filter $\{g_{k,j}\}$, that form a (frame) multiwavelet.

As in the classical case, one must be clever in choosing the filter $\{h_{i,j}\}$. Also, as mentioned above, the construction of the complementary functions $\{g_{k,j}\}$ is not as routine as in the classical case. In Section 4, we give some concrete examples of how our techniques work. That is, we construct some generalized conjugate mirror filters with desirable properties, and then use them to build generalized scaling functions and wavelet (frames). As mentioned above, among our examples here we include a non-MRA wavelet in $L^2(\mathbb{R})$ whose Fourier transform is C^∞ on an arbitrarily large interval.

Some earlier constructions of wavelets from filters in the special case of a single generalized scaling function (a *frame multiresolution analysis*) can be found in [BL], [PSWX], [PSW], and [Han]. See the end of Section 2 of this paper for some further discussion of FMRAs. Similar constructions in the special case of a constant multiplicity function appear in [JS] and [HLPS]. We note that even in these special cases, our procedure differs from this previous work in that we begin our constructions using only a multiplicity function rather than using variations on known scaling functions or filters. The authors owe a special thanks to Robert Strichartz, whose exposition of the classical constructions in [Str1] and [Str2] provided insights helpful to our generalizations.

2. FILTERS AND WAVELETS FROM GMRAs

We let the group $\Gamma = \mathbb{Z}^n$, act on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$, by

$$[\gamma(f)](x) = f(x+\gamma).$$

Given an expansive $n \times n$ integer matrix A with $|\det A| = d$, we define another unitary operator δ by

$$[\delta(f)](x) = \sqrt{d} f(A(x)).$$

By expansive, we mean that all the eigenvalues of A have absolute value greater than 1. This implies that there is a neighborhood F of the origin in $\mathbb{R}^n$ that satisfies $F \subset A^*F$ and $\cup_{j\geq 0} A^{*j}F = \mathbb{R}^n$, where A^* denotes the transpose of A. (See e.g. [GH].) We will make use of the fact that the map $\gamma \mapsto \delta^{-1}\gamma\delta = A\gamma$ is an isomorphism of $\mathbb{Z}^n$ onto the proper subgroup $A\mathbb{Z}^n$, which has index $d = |\det A| > 1$ in $\mathbb{Z}^n$. This isomorphism induces a homomorphism α on the dual group $\widehat{\Gamma} = [-\pi, \pi)^n$ defined by $\alpha(\omega)(\gamma) = \omega(\delta^{-1}\gamma\delta)$. The homomorphism α sends ω to $A^*\omega$ (mod 2π), and when there is no ambiguity, we will refer to it as A^*. The matrix A^* also gives an isomorphism of $\mathbb{Z}^n$ onto $A^*\mathbb{Z}^n$. We will have need for a set of coset representatives of $\mathbb{Z}^n/A^*\mathbb{Z}^n$, which we label as $l_0, l_1, \cdots l_{d-1}$, with $l_0 = 0$. For $\omega \in [-\pi, \pi)^n$, we then let $\omega_q = A^{*-1}(\omega + 2\pi l_q)$, so that $\omega_0, \omega_1, \cdots \omega_{d-1}$ are the d pre-images of ω under α, with $\omega_0 = A^{*-1}\omega$. We will consistently use Haar measure $\frac{1}{(2\pi)^n} d\omega$ on $[-\pi, \pi)^n$, Lebesgue measure on $\mathbb{R}^n$, and Fourier transform given by $\widehat{f}(\xi) = \int f(y) e^{-i\langle \xi | y \rangle} dy$. In the context of the actions described above of the unitary operators given by Γ and δ on $\mathcal{H} = L^2(\mathbb{R}^n)$, we make the following definitions:

DEFINITION. A (orthonormal) *multiwavelet* for $\mathcal{H}$ relative to Γ and δ is a collection $\{\psi_1, \psi_2, \ldots\}$ of vectors in $\mathcal{H}$ such that the collection $\{\delta^j(\gamma(\psi_i))\}$, for $j \in \mathbb{Z}$, $\gamma \in \Gamma$, and $i \geq 1$, forms an orthonormal basis for $\mathcal{H}$. If the collection $\{\delta^j(\gamma(\psi_i))\}$ forms instead only a normalized tight frame for $\mathcal{H}$, it is called a *frame multiwavelet.*

Every multiwavelet determines a more general structure of nested subspaces of $\mathcal{H}$ by setting V_j equal to the closure of the span of the vectors $\delta^k(\gamma(\psi_i))$, for $\gamma \in \Gamma$, $i \geq 1$, and $k < j$. These subspaces satisfy the definition below ([BMM]):

DEFINITION. A *generalized multiresolution analysis* (GMRA) of $\mathcal{H}$, relative to Γ and δ, is a collection $\{V_j\}_{-\infty}^{\infty}$ of closed subspaces of $\mathcal{H}$ that satisfy:

(1) $V_j \subseteq V_{j+1}$ for all j.
(2) $\delta(V_j) = V_{j+1}$ for all j.
(3) $\cup V_j$ is dense in $\mathcal{H}$ and $\cap V_j = \{0\}$.
(4) V_0 is invariant under the action of Γ.

A GMRA also determines a mutually orthogonal sequence of subspaces W_j, defined by $V_{j+1} = V_j \oplus W_j$, whose closed linear span is $\mathcal{H}$.

Unlike the classical definition of a *multiresolution analysis* (MRA), a GMRA does not require the existence of a scaling vector ϕ whose translates form an orthonormal basis for V_0. However, as shown in [BMM], and [BM], we can obtain similar information about its structure by studying the unitary representation determined by the action of Γ on V_0. By the spectral multiplicity theory developed by Stone ([Sto]) and Mackey ([Mac]) (see also [Hal] and [Hel]), the representation is completely determined by a multiplicity function m mapping $\widehat{\Gamma}$ into the set $\{0, 1, 2, \ldots, \infty\}$. (In general, a complete description of the representation also requires specifying a measure class, but in the case of a GMRA on $L^2(\mathbb{R}^n)$, the measure must always be absolutely continuous with respect to Lebesgue measure [BMM].) The multiplicity function roughly counts the number of times each character in $\widehat{\Gamma}$ occurs in the representation. If the GMRA is an MRA, translates by Γ of the scaling function ϕ give an orthonormal basis for V_0. Thus in this case, the representation of translation by Γ on V_0 is equivalent to the regular representation of Γ. The regular representation is known to (weakly) contain every character exactly once, so in the MRA case we have $m \equiv 1$. Many other multiplicity functions are possible for GMRA's. (See [BMM], and [BM].)

To use the information the multiplicity function provides about ρ, we form the direct sum $L^2(S_1) \oplus L^2(S_2) \oplus \cdots$, where $S_j = \{x \in [-\pi, \pi)^n : m(x) \geq j\}$. The properties of m guarantee the existence of a unitary map $J : V_0 \mapsto \oplus_{j=1}^{\infty} L^2(S_j)$ which intertwines the actions of Γ on V_0 and on $\oplus_{j=1}^{\infty} L^2(S_j)$. (See [Cou] for more details.)

The following Theorem summarizes the properties of the multiplicity function m and the map J that we will need in this paper.

THEOREM 2.1. *Let $\{V_j\}$ be a GMRA in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ relative to unitary operators given by Γ and δ. Then there exists an (almost everywhere) unique function $m : [-\pi, \pi)^n \to \{0, 1, 2, \ldots, \infty\}$ such that:*

(1) *For each $j > 0$, set $S_j = \{\omega \in [-\pi, \pi)^n : m(\omega) \geq j\}$. Then there exists a (not necessarily unique) unitary operator $J : V_0 \to \bigoplus_{j=1}^{\infty} L^2(S_j)$ satisfying*

$$[J(\gamma(f))](\omega) = e^{i\langle\gamma|\omega\rangle}[J(f)](\omega)$$

for all $\gamma \in \mathbb{Z}^n$, all $f \in V_0$, and μ almost all $\omega \in [-\pi, \pi)^n$.

(2) *Let J be as in (1). Let χ_i be the element of $\bigoplus_{j=1}^{\infty} L^2(S_j)$ whose ith component is χ_{S_i}, the characteristic function of S_i, and whose other components are all 0; set $\phi_i = J^{-1}(\chi_i)$. Then the collection $\{\gamma(\phi_i)\}$ forms a normalized tight frame for V_0.*

(3) *The function m satisfies the following consistency inequality:*

$$m(\omega) \leq \sum_{l=0}^{d-1} m(\omega_l)$$

for almost all ω, where $\omega_0, \omega_2, \cdots \omega_{d-1}$ are the d points in $[-\pi, \pi)^n$ such that $A^*\omega_l = \omega \bmod 2\pi$.

(4) *There exist vectors $\psi_1, \psi_2 \cdots$ in the subspace W_0 that form a frame multiwavelet for $L^2(\mathbb{R}^n)$. These vectors form an orthonormal N-wavelet if and only if $\left(\sum_{l=0}^{d-1} m(\omega_l)\right) - m(\omega) = N$ a.e..*

PROOF. See [BMM] and [BM].

The function m of Theorem 2.1 is called the *multiplicity function* associated to the GMRA $\{V_j\}$. Any set of vectors $\{\phi_i\}$ in V_0 that satisfy the frame condition in

(2) are called *generalized scaling vectors*. In general, the multiplicity function m may take on the value ∞ on a set of positive measure. However, in this paper, we will consider only the case where m is finite almost everywhere. In this case, we can define $\widetilde{m}(\omega) = \left(\sum_{l=0}^{d-1} m(\omega_l)\right) - m(\omega)$, so that (3) of Theorem 2.1 can be rewritten as the following *consistency equation:*

$$m(\omega) + \widetilde{m}(\omega) = \sum_{l=0}^{d-1} m(\omega_l).$$

With this definition, (4) of Theorem 2.1 states that if $\widetilde{m}$ has constant value N, the GMRA has an associated orthonormal N-wavelet; if the value of $\widetilde{m}$ is not constant, the GMRA has only a frame multiwavelet. Analogous to the definitions of sets S_j associated to m, we let $\widetilde{S}_j = \{\omega : \widetilde{m}(\omega) \geq j\}$.

In the classical case of dilation by 2 in $L^2(\mathbb{R})$, Mallat and Meyer constructed wavelets using a tool called a conjugate mirror filter (CMF) associated to the scaling function ϕ of an MRA. Following the work of Courter [Cou], we will show that the generalized scaling vectors defined above yield a collection of functions analogous to the CMF of Mallat and Meyer. First we need the following technical result:

LEMMA 2.2. *Let $f \in L^2([-\pi,\pi)^n)$, and define $\widetilde{f}(\omega) = \sum_{l=0}^{d-1} f(\omega_l)$. Then the Fourier coefficients of $\widetilde{f}$ and f satisfy*

$$c_k(\widetilde{f}) = dc_{Ak}(f)$$

for all $k \in \mathbb{Z}^n$, where $\omega_0, \omega_2, \cdots \omega_{d-1}$ are the d points in $[-\pi,\pi)^n$ such that $A^*\omega_l = \omega$ mod 2π.

PROOF. Let $z_l = \omega_l - \omega_0$. Then $z_0 = 0, z_1, \cdots z_{d-1}$ are the d elements of the kernel of the homomorphism $\alpha = A^*$ acting on $[-\pi,\pi)^n$. Thus, making the change of variables $y = \omega_l$, we get

$$\begin{aligned}
c_k(\widetilde{f}) &= \frac{1}{(2\pi)^n} \int_{[-\pi,\pi)^n} \sum_{l=0}^{d-1} f(\omega_l) e^{-i\langle k|\omega\rangle} d\omega \\
&= \frac{1}{(2\pi)^n} \sum_{l=0}^{d-1} \int_{[-\pi,\pi)^n} f(\omega_l) e^{-i\langle Ak|\omega_l\rangle} d\omega \\
&= \frac{d}{(2\pi)^n} \sum_{l=0}^{d-1} \int_{A^{*-1}[-\pi,\pi)^n + z_l} f(y) e^{-i\langle Ak|y\rangle} dy \\
&= \frac{d}{(2\pi)^n} \int_{[-\pi,\pi)^n} f(y) e^{-i\langle Ak|y\rangle} dy \\
&= dc_{Ak}(f).
\end{aligned}$$

We are now ready to prove the existence of a generalized scaling function analog to the conjugate mirror filters of Mallat and Meyer.

DEFINITION. Let m be an arbitrary measurable function mapping $[-\pi,\pi)^n$ into $\{0,1,2,\cdots\}$ that satisfies the consistency inequality of Theorem 2.1, and let $S_j = \{\omega : m(\omega) \geq j\}$. A set $\{h_{i,j}\}$ of functions on $[-\pi,\pi)^n$, with support$(h_{i,j}) \subset S_j$, and satisfying

$$\sum_j \sum_{l=0}^{d-1} h_{i,j}(\omega_l)\overline{h_{k,j}(\omega_l)} = \begin{cases} d\chi_{S_i}(\omega) & i = k \\ 0 & i \neq k \end{cases}$$

for almost all $\omega \in [-\pi,\pi)^n$, is called a *generalized conjugate mirror filter* (GCMF) relative to m.

Remark.. In the definition of GCMF, i and j take on all nonzero values in the range of m. We will sometimes write the GCMF in the form $h_i = \oplus_j h_{i,j} \in \bigoplus_{j=1}^{\infty} L^2(S_j)$. We will also use the notation $\{h_{i,j}\}$ for the 2π periodic extensions of the GCMF $\{h_{i,j}\}$.

THEOREM 2.3. *Let J, $\{\phi_i\}$, m and S_i be as in Theorem 2.1. Define functions $h_{i,j}$ on $[-\pi,\pi)^n$ by the condition*

$$J(\delta^{-1}(\phi_i)) = h_i = \bigoplus_j h_{i,j}.$$

Then the functions $h_{i,j}$ are a GCMF relative to m.

PROOF. Fix i and k, and let $f(\omega) = \sum_j h_{i,j}(\omega)\overline{h_{k,j}(\omega)}$. Then, using the notation of Lemma 2.2, we must show

$$\widetilde{f}(\omega) = \begin{cases} d\chi_{S_i}(\omega) & i = k \\ 0 & i \neq k \end{cases}.$$

We have for $\gamma \in \mathbb{Z}^n$

$$\begin{aligned} c_\gamma(\widetilde{f}) &= dc_{A\gamma}(f) \\ &= d\langle h_i \mid e^{i\langle A\gamma|\cdot\rangle} h_k\rangle_{\bigoplus L^2(S_j)} \\ &= d\langle \delta^{-1}\phi_i \mid A\gamma\delta^{-1}\phi_k\rangle_{\mathcal{H}} \\ &= d\langle \phi_i \mid \gamma\phi_k\rangle_{\mathcal{H}} \\ &= d\langle \chi_i \mid e^{i\langle\gamma|\cdot\rangle}\chi_k\rangle_{\bigoplus L^2(S_j)} \\ &= \begin{cases} dc_\gamma(\chi_{S_i}) & i = k \\ 0 & i \neq k \end{cases}. \end{aligned}$$

We will continue to mimic the construction technique used by Mallat and Meyer in the classical MRA setting. After identifying a CMF associated to the scaling function of the MRA, Mallat and Meyer used it to build a second CMF, and then showed that this second CMF must be associated to a wavelet. Following Courter [Cou], we develop the following analog for GMRAs.

DEFINITION. Let $m : \widehat{\Gamma} \to \{0,1,2,\ldots\}$ be an arbitrary measurable function that satisfies the consistency inequality of Theorem 2.1, and let $\{h_{i,j}\}$ be a GCMF relative to m. As before, let $\widetilde{m}(\omega) = \left(\sum_{l=0}^{d-1} m(\omega_l)\right) - m(\omega)$, and $\widetilde{S}_j = \{\omega : \widetilde{m}(\omega) \geq j\}$. By a *complementary conjugate mirror filter* (CCMF), we will mean a collection $\{g_{k,j}\}$ of functions on $[-\pi,\pi)^n$, with support$(g_{k,j}) \subset S_j$, which satisfy:

$$\sum_j \sum_{l=0}^{d-1} g_{k',j}(\omega_l)\overline{g_{k,j}(\omega_l)} = \begin{cases} d\chi_{\widetilde{S}_k}(\omega) & k = k' \\ 0 & k \neq k' \end{cases}$$

and

$$\sum_j \sum_{l=0}^{d-1} h_{i,j}(\omega_l)\overline{g_{k,j}(\omega_l)} = 0$$

for all i and k.

Remark. In the definition of a CCMF, k takes on nonzero values in the range of $\widetilde{m}$, while j takes on nonzero values in the range of m. As with GCMFs, we will sometimes write a CCMF in the form $g_k = \oplus_j g_{k,j} \in \bigoplus_{j=1}^{\infty} L^2(S_j)$. We will again use the notation $\{g_{k,j}\}$ as well for the 2π periodic extensions of the CCMF $\{g_{k,j}\}$.

The construction of CCMFs from GCMFs relies on linear algebra arguments developed in [Cou] that generalize Mallat and Meyer's constructions to allow the dimension of the vectors built from the filters to change with ω. We adopt the following notation. Fix an $\omega \in [-\pi,\pi)^n$, and again write $\omega_0, \omega_1, \cdots \omega_{d-1}$ for the d points in $[-\pi,\pi)^n$ such that $A^*(\omega_j) = \omega \bmod 2\pi$. Let $l_j = m(\omega_j)$ for $0 \leq j \leq d-1$. Note that $\omega_k \notin S_j$ for any $j > l_k$, and by the consistency equation, we have that $\sum_{j=0}^{d-1} l_j = m(\omega) + \widetilde{m}(\omega)$. Let $f = \oplus f_j \in \bigoplus_{j=1}^{\infty} L^2(S_j)$. Since f_j is supported on S_j and $\omega_k \notin S_j$ for $j > m(\omega_k)$, we know that $f_j(\omega_k) = 0$ for $j > m(\omega_k) = l_k$. Thus we can build a vector $\vec{f^\omega}$ whose components include all of the nonzero values of the f_j at the points ω_k as follows:

$$\vec{f^\omega} = (f_1(\omega_0), \cdots, f_{l_0}(\omega_0), f_1(\omega_1), \cdots, f_{l_1}(\omega_1), \cdots, f_1(\omega_{d-1}), \cdots, f_{l_{d-1}}(\omega_{d-1})).$$

For each fixed ω, the consistency equation shows that $\vec{f^\omega}$ is a vector of length $m(\omega) + \widetilde{m}(\omega)$. If $f, g \in \bigoplus_{j=1}^{\infty} L^2(S_j)$, we will write $\langle \vec{f^\omega} \mid \vec{g^\omega} \rangle$ and $\|\vec{f^\omega}\|$ for the ordinary inner product and norm in $\mathbb{C}^{m(\omega)+\widetilde{m}(\omega)}$. Thus

$$\langle \vec{f^\omega} \mid \vec{g^\omega} \rangle = \sum_j \sum_{r=0}^{d-1} f(\omega_r)\overline{g(\omega_r)}.$$

With this notation, the equations in the definitions of GCMF and CCMF can be written as follows.

LEMMA 2.4. *The functions $\{h_{i,j}\}$ are a GCMF relative to the multiplicity function m if and only if* support$(h_{i,j}) \subset S_j$ *and*

$$\langle \vec{h}_i^\omega \mid \vec{h}_k^\omega \rangle = \begin{cases} d\chi_{S_i}(\omega) & i = k \\ 0 & i \neq k \end{cases}$$

for almost all ω. The functions $\{g_{k,j}\}$ are an associated CCMF if and only if $\operatorname{support}(g_{k,j}) \subset S_j$ *and*

$$\langle \vec{g}_i^{\omega} \mid \vec{g}_j^{\omega} \rangle = \begin{cases} d\chi_{\widetilde{S}_i}(\omega) & i = j \\ 0 & i \neq j \end{cases},$$

and

$$\langle \vec{g}_i^{\omega} \mid \vec{h}_j^{\omega} \rangle = 0$$

for almost all ω and all i and j.

THEOREM 2.5. *Let $\{V_j\}$ be a GMRA in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$, with multiplicity function m that is finite almost everywhere. Suppose $\{h_{i,j}\}$ is a GCMF constructed as in Theorem 2.3. Then*

1. *A CCMF $\{g_{k,j}\}$ can be constructed explicitly from $\{h_{i,j}\}$.*
2. *If $\{g_{k,j}\}$ is any CCMF for $\{h_{i,j}\}$, and we set $\psi_k = \delta(J^{-1}(\oplus g_{k,j}))$, then the collection $\{\delta^j(\gamma(\psi_k))\}$, for $\gamma \in \Gamma$, $j \in \mathbb{Z}$, and $k > 0$, forms a frame multiwavelet for $\mathcal{H}$.*
3. *Suppose $\widetilde{m} \equiv N$, and let $\{g_{k,j}\}$ be any CCMF for $\{h_{i,j}\}$. If we set $\psi_k = \delta(J^{-1}(\oplus g_{k,j}))$, then the collection $\{\delta^j(\gamma(\psi_k))\}$, for $\gamma \in \Gamma$, $j \in \mathbb{Z}$, and $1 \leq k \leq N$, forms a multiwavelet for $\mathcal{H}$.*

PROOF. Proof of (1): Partition $[-\pi, \pi)^n$ into a countable collection of sets of the form $P_{l_0,l_1,\cdots l_{d-1},\widetilde{l}} = \{\omega \in [-\pi,\pi)^n : m(\omega_r) = l_r \text{ and } \widetilde{m}(\omega) = \widetilde{l}\}$. We will define the functions $g_{i,j}$, $i \geq 1$, piecewise on the sets $P_{l_0,l_1,\cdots l_{d-1},\widetilde{l}}$ by fixing an $\omega \in P_{l_0,l_1,\cdots l_{d-1},\widetilde{l}}$, and building vectors $\vec{g}_r^{\omega}$ (as described above Lemma 2.4) measurably in ω. For each fixed ω, the GCMF $\{h_{i,j}\}$ determines $m(\omega)$ orthogonal vectors $\vec{h}_i^{\omega}$ in $\mathbb{C}^{m(\omega)+\widetilde{m}(\omega)}$. To produce the CCMF, we must measurably construct $\widetilde{m}(\omega)$ more orthogonal vectors $\vec{g}_r^{\omega} \in \mathbb{C}^{m(\omega)+\widetilde{m}(\omega)}$ each of norm $\sqrt{d}$. This is always possible by the obvious dimension argument. It can be accomplished by inspection, by Gram-Schmidt, or by the following generalization of Courter's technique for the case $\widetilde{m}(\omega) \equiv 1$ [Cou].

If $\widetilde{m}(\omega) \geq 1$ we can find $m(\omega)+1$ components of the orthogonal vectors $\vec{h}_i^{\omega}$ such that if we only look at these components, they form $m(\omega)$ independent vectors in $\mathbb{C}^{m(\omega)+1}$ space. Take a cross product of these $m(\omega)$ vectors in $\mathbb{C}^{m(\omega)+1}$ to get an orthogonal vector in $\mathbb{C}^{m(\omega)+1}$. Finally, enlarge this vector to a vector in $\mathbb{C}^{m(\omega)+\widetilde{m}(\omega)}$ by inserting 0's in missing components. The resulting vector, properly normalized, is our first $\vec{g}_1^{\omega}$. We then repeat the process to get the rest of the $\vec{g}_r^{\omega}$.

Proof of (2): Let $T = \{g_1, h_1, g_2, h_2, \cdots\} \subset \bigoplus_{j=1}^{\infty} L^2(S_j)$. First notice that for any fixed ω, Lemma 2.4 shows that the vectors $\frac{1}{\sqrt{d}}\vec{g}_i^{\omega}$ for $1 \leq i \leq \widetilde{m}(\omega)$ and $\frac{1}{\sqrt{d}}\vec{h}_j^{\omega}$ for $1 \leq j \leq m(\omega)$ form an orthonormal basis of $\mathbb{C}^{m(\omega)+\widetilde{m}(\omega)}$. We use this to show that $\{e^{i\langle Ak|\cdot\rangle}\tau : \tau \in T, k \in \mathbb{Z}^n\}$ form a normalized tight frame for $\bigoplus_{j=1}^{\infty} L^2(S_j)$. Let $f = \oplus f_j$ be an arbitrary element of $\bigoplus_{j=1}^{\infty} L^2(S_j)$. Then, using Lemma 2.2 we see that

$$\begin{aligned}
\sum_{\tau,k} |\langle f \mid e^{i\langle Ak|\cdot\rangle}\tau\rangle_{\bigoplus_{j=1}^{\infty} L^2(S_j)}|^2 &= \sum_{\tau,k} |c_{Ak}(\sum_j f_j\bar{\tau}_j)|^2 \\
&= \frac{1}{d^2}\sum_{\tau,k} |c_k(\langle \vec{f^\omega} \mid \vec{\tau^\omega}\rangle)|^2 \\
&= \frac{1}{d^2}\sum_{\tau} \|\langle \vec{f^\omega} \mid \vec{\tau^\omega}\rangle\|^2_{L^2([-\pi,\pi)^n)} \\
&= \frac{1}{(2\pi)^n d^2}\sum_{\tau}\int_{[-\pi,\pi)^n} |\langle \vec{f^\omega} \mid \vec{\tau^\omega}\rangle|^2 d\omega \\
&= \frac{1}{d(2\pi)^n}\int_{[-\pi,\pi)^n} \|\vec{f^\omega}\|^2_{\mathbb{C}^{m(\omega)+\widetilde{m}(\omega)}} d\omega \\
&= \frac{1}{d} c_0(\sum_j \sum_{l=0}^{d-1} |f_j(\omega_l)|^2) \\
&= c_{A(0)}(\sum_j |f_j|^2) \\
&= \|f\|^2_{\bigoplus_{j=1}^{\infty} L^2(S_j)}
\end{aligned}$$

It follows immediately from (1) of Theorem 2.1 that the functions $(A\gamma)\delta^{-1}\psi_i = \delta^{-1}\gamma\psi_i$ and $(A\gamma)\delta^{-1}\phi_i = \delta^{-1}\gamma\phi_i$, for $\gamma \in \mathbb{Z}^n$, form a normalized tight frame for V_0. Since δ is unitary, this shows that the functions $\gamma\psi_i, \gamma\phi_i$ for $\gamma \in \mathbb{Z}^n$ form a normalized tight frame for V_1. Writing $V_1 = V_0 \oplus W_0$, it follows from (2) of Theorem 2.1 that $\{\gamma\psi_i\}$ form a normalized tight frame for W_0, and thus that $\{\delta^j\gamma\psi_i\}$ form a tight frame for $\mathcal{H}$.

Proof of (3): If $\widetilde{m} \equiv N$, we have $\widetilde{S}_1, \widetilde{S}_2, \cdots \widetilde{S}_N$ all equal to $[-\pi,\pi)^n$, and $\widetilde{S}_j = \emptyset$ if $j > N$. Thus, for almost all $\omega \in [-\pi,\pi)^n$, the vectors $\frac{1}{\sqrt{d}}\vec{g}_i^\omega$ for $1 \le i \le N$ form an orthonormal basis for $\mathbb{C}^N$. A similar calculation to that of part 2 shows

$$\begin{aligned}
\langle e^{i\langle Ak|\cdot\rangle} g_i \mid e^{i\langle Ak'|\cdot\rangle} g_{i'}\rangle &= c_{A(k'-k)}(\sum_j g_{i,j}\bar{g}_{i',j}) \\
&= \frac{1}{d} c_{k-k'}(\langle \vec{g}_i^\omega \mid \vec{g}_{i'}^\omega\rangle) \\
&= \begin{cases} 1 \text{ if } k = k' \text{ and } i = i' \\ 0 \qquad\qquad \text{otherwise} \end{cases}
\end{aligned}$$

so that the functions $e^{i\langle Ak|\cdot\rangle} g_i$ for $k \in \mathbb{Z}$ and $1 \le i \le N$ form an orthonormal basis for their span in $\bigoplus_{j=1}^{\infty} L^2(S_j)$. Again, following the argument in part (2) we see that the functions $\gamma\psi_i$ for $\gamma \in \mathbb{Z}$ and $1 \le i \le N$ form an orthonormal basis of W_0 and thus that the functions $\delta^j\gamma\psi_i$ form an orthonormal basis for $\mathcal{H}$.

We close this section by specializing the development above to the case of an FMRA, which is a GMRA $\{V_j\}$ for which there is a single generalized scaling function $\phi \in V_0$. From our point of view, this is equivalent to the condition that the multiplicity function m associated to the GMRA is simply the characteristic function χ_S of some subset $S \subseteq \widehat{\Gamma}$. Since in this case m only takes on the values 0 and 1, it follows that any GCMF relative to m will be a single function h, and the mirror equation it satisfies is

$$\sum_{l=0}^{d-1} |h(\omega_l)|^2 = d\chi_S(\omega).$$

We note that this is different from the "mirror equations" for FMRAs studied in [PSW].

The multiplicity function $m \equiv \chi_S$ satisfies the consistency equation, showing that the associated function $\widetilde{m}$ could very well take on values between 0 and d. Therefore, there may be as many as d elements in the corresponding frame multiwavelet.

In Section 4 we will use our general techniques to construct explicitly an FMRA and its associated frame multiwavelet. In the classical case of dilation by 2 in $L^2(\mathbb{R})$, we are able to do this in such a way that the Fourier transform of the wavelet is C^∞ on an arbitrarily large interval.

3. MULTIPLICITY FUNCTIONS

In the last section, we showed how to build (frame) multiwavelets from the GCMF associated to the generalized scaling functions of GMRAs. The missing ingredient from this formula is a technique for finding GMRAs and their generalized scaling functions. In this Section, we show how to build generalized scaling functions and thus GMRAs and wavelets from simple GCMFs generalized from the $L^2(\mathbb{R})$ case. We are able to do this starting only with knowledge of a multiplicity function m. To begin this construction, we need the following determination of exactly which functions are multiplicity functions for GMRAs. As before, given a function $m : [-\pi,\pi)^n \mapsto \{0,1,2,\cdots\}$, we let $S_i = \{\omega \in [-\pi,\pi)^n : m(\omega) \geq i\}$. Let $\Delta = \cap_{k=0}^{\infty} A^{*k}(S_1 + 2\pi\mathbb{Z}^n)$.

PROPOSITION 3.1. *If $m : [-\pi,\pi)^n \mapsto \{0,1,2,\cdots\}$ is an integrable function such that*

(1) *m satisfies the consistency inequality*

$$m(\omega) \leq \sum_{l=0}^{d-1} m(\omega_l)$$

(2)
$$\sum_{\gamma\in\Gamma} \chi_\Delta(\omega + 2\pi\gamma) \geq m(\omega).$$

(3)
$$\cup_{p\in\mathbb{Z}} A^{*p}\Delta = \mathbb{R}^n$$

.

Then m is a multiplicity function for a GMRA in $L^2(\mathbb{R}^n)$. In this case there exists a generalized scaling set $E = \cup_{i=1}^{\infty} E_i \subset \mathbb{R}^n$ for m with the following properties:

(i) The sets E_i are disjoint, with E_i congruent to S_i mod 2π.

(ii) $E_k \subseteq A^*(\cup_{j=1}^{k} E_j)$.

(iii) If there exists a neighborhood of the origin on which $m(\omega) > 0$, then E_1 contains a neighborhood of 0 and $\cup_{j=k}^{\infty} A^{*j} E_1 = \mathbb{R}^n$ for every $k \in \mathbb{Z}$.

(iv) If $S_1 \subset A^* S_1$ then $E_1 = S_1$.

PROOF. The sufficiency of the three conditions listed for an integrable function to be a multiplicity function for a GMRA is established in Theorem 1.5 of [BM]. (See [BRS] for a similar result.) The proof of Theorem 1.5 of [BM] constructs the generalized scaling set E with properties (i), (ii) and (iv) above. Property (iii) follows from the fact that when the support of m contains a neighborhood of the origin, the expansive properties of A imply that Δ will also contain a (possibly different) neighborhood. By the first stage of the construction of E in [BM] , this guarantees that E_1 too will contain a neighborhood of the origin. Property (ii) and the fact that A is expansive then show that $\cup_{j=k}^{\infty} A^{*j} E_1 = \mathbb{R}^n$ for every $k \in \mathbb{Z}$.

We will use this information about multiplicity functions, particularly the existence of the scaling set $E = \cup E_i$, to show how to build GMRAs using GCMFs. We will do this by constructing the Fourier transforms of generalized scaling functions. We need the following tool, which is analogous to a well-known MRA result (see e.g. [HW] p. 382). As in the MRA result, the three conditions on the $\{\hat{\phi}_i\}$ in Theorem 3.2 are actually both necessary and sufficient. For brevity, we include only the direction we need.

THEOREM 3.2. *Suppose m is an integrable function on the cube $[-\pi, \pi)^n$ satisfying the hypotheses of Proposition 3.1, with $S_i = \{\omega \in [-\pi,\pi)^n : m(\omega) \geq i\}$, and that the collection $\{h_{i,j}\}$ is a GCMF relative to m. Suppose that $\{\phi_i\}$ is a collection of functions in $L^2(\mathbb{R}^n)$ that satisfy the following three conditions.*

(1) *For almost all $\omega \in [-\pi, \pi)^n$, we have*

$$\sum_{l \in \mathbb{Z}^n} \widehat{\phi}_i(\omega + 2\pi l)\overline{\widehat{\phi}_j}(\omega + 2\pi l) = \begin{cases} 0 & \text{for } i \neq j \\ \chi_{S_i}(\omega) & \text{for } i = j \end{cases}.$$

(2) *For each i we have*

$$\widehat{\phi}_i(A^*(\xi)) = \frac{1}{\sqrt{d}} \sum_j h_{i,j}(\xi)\widehat{\phi_j}(\xi)$$

for almost all $\xi \in \mathbb{R}^n$.

(3) *For almost all $\xi \in \mathbb{R}^n$,*

$$\limsup_{j \to \infty} \sum_i |\widehat{\phi}_i(A^{*-j}(\xi))|^2 \geq 1.$$

Then the ϕ_i's are generalized scaling functions for a GMRA $\{V_j\}$ whose associated multiplicity function is the given function m.

PROOF. Set V_0 equal to the closure of the span of the integral translates of the ϕ_i's, and set V_j equal to $\delta^j(V_0)$. We will show that (1) implies that the translates of the ϕ_i's form a normalized tight frame for V_0, and that the representation of $\mathbb{Z}^n$ on V_0 has associated multiplicity function equal to m. We will show next that (2) implies that $V_0 \subseteq \delta(V_0)$ so that $V_j \subseteq V_{j+1}$. The fact that $\cap V_j = \{0\}$ will follow as a consequence of the integrability of m together with (1), while the denseness of $\cup V_j$ will follow from (3).

If $f \in V_0$, then $\widehat{f}$ must be of the form $\sum_j u_j \widehat{\phi_j}$, where each u_j is periodic. We have then, using (1), the following calculation for every $f \in V_0$.

$$\begin{aligned}
\sum_i \sum_\gamma |\langle f \mid \gamma(\phi_i)\rangle|^2 &= \sum_i \sum_\gamma \frac{1}{(2\pi)^{2n}} \Big| \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{\phi_i}}(\xi) e^{-i\langle\gamma|\xi\rangle}\, d\xi\Big|^2 \\
&= \frac{1}{(2\pi)^{2n}} \sum_i \sum_\gamma \Big| \int_{[-\pi,\pi)^n} \sum_j u_j(\omega) \sum_l \widehat{\phi_j}(\omega + 2\pi l) \\
&\qquad \times \overline{\widehat{\phi_i}}(\omega+2\pi l) e^{-i\langle\gamma|\omega\rangle}\, d\omega\Big|^2 \\
&= \frac{1}{(2\pi)^{2n}} \sum_i \sum_\gamma \Big| \int_{[-\pi,\pi)^n} u_i(\omega)\chi_{S_i}(\omega) e^{-i\langle\gamma|\omega\rangle}\, d\omega\Big|^2 \\
&= \frac{1}{(2\pi)^n} \sum_i \int_{[-\pi,\pi)^n} |u_i(\omega)|^2 \chi_{S_i}(\omega)\, d\omega \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_i |u_i(\xi)\widehat{\phi_i}(\xi)|^2\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2\, d\xi \\
&= \|f\|^2,
\end{aligned}$$

so that the functions $\{\gamma(\phi_i)\}$ form a normalized tight frame for V_0.

Now define an operator $J : V_0 \to \bigoplus L^2(S_i)$ by the formula

$$[J(f)]_i(\omega) = \sum_\gamma \langle f \mid \gamma(\phi_i)\rangle e^{i\langle\gamma|\omega\rangle} \chi_{S_i}(\omega).$$

One sees directly that this operator J is unitary and satisfies the intertwining condition of Theorem 2.1 (1). In particular, the multiplicity function associated to the representation of $\mathbb{Z}^n$ on V_0 is equivalent to the multiplicity function associated to the representation of $\mathbb{Z}^n$ acting by multiplication by exponentials on the direct sum $\bigoplus_i L^2(S_i)$, and this multiplicity function is $\sum_i \chi_{S_i}$, which is the given function m.

Next, we write the periodic function $h_{i,j}$ in its Fourier series,

$$h_{i,j}(\omega) = \sum_\gamma c_{i,j,\gamma} e^{i\langle\gamma|\omega\rangle},$$

so that we see by (2) that

$$\begin{aligned}\sqrt{d}\widehat{\phi}_i(A^*(\xi)) &= \sum_j h_{i,j}(\xi)\widehat{\phi_j}(\xi) \\ &= \sum_j \sum_\gamma c_{i,j,\gamma} e^{i\langle\gamma|\xi\rangle}\widehat{\phi}_j(\xi)\end{aligned}$$

so that,

$$\delta^{-1}(\phi_i) = \sum_j \sum_\gamma c_{i,j,\gamma}\gamma(\phi_j).$$

Hence, $\delta^{-1}(\phi_i)$ belongs to the closed linear span of the translates of the ϕ_j's. So, we have that $V_0 \subseteq \delta(V_0)$, as desired.

Write P_j for the orthogonal projection operator onto the subspace V_j. To prove that $\cap V_j = \{0\}$, it will suffice to show that $\lim_{j\to\infty} \|P_{-j}(f)\| = 0$ for each $f \in L^2(\mathbb{R}^n)$. By a standard approximation argument, it will suffice to show this holds on a dense subset of $L^2(\mathbb{R}^n)$. Thus, let f be a Schwartz function for which $\widehat{f}$ vanishes in some neighborhood N_f of 0, and write C_f for the (finite) number $\sum_k |f * f^*(k)|$. Such f's are dense in $L^2(\mathbb{R}^n)$. The Poisson Summation Formula holds for such an f, and we will use it in the following form:

$$\frac{1}{d^j}\sum_l |\widehat{f}(A^{*-j}(\xi + 2\pi l))|^2 = \sum_k f * f^*(A^j(k))e^{-i\langle k|\xi\rangle}.$$

Now, for each $\xi \in [-\pi,\pi)^n$, let $l_j(\xi)$ be the largest number for which $A^{*-j}(\xi+2\pi l) \in N_f$ for all $|l| < l_j(\xi)$. Because A is expansive, we must have that $l_j(\xi)$ tends to infinity for almost every ξ. Finally, we use the fact that the function

$$m(\omega) = \sum_i \chi_{S_i}(\omega) = \sum_i \sum_l |\widehat{\phi}_i(\omega + 2\pi l)|^2$$

is assumed to be integrable on the cube $[-\pi,\pi)^n$. Hence, we have

$$
\begin{aligned}
\|P_{-j}(f)\|^2 &= \frac{d^j}{(2\pi)^{2n}} \sum_i \sum_\gamma \Big| \int_{\mathbb{R}^n} \widehat{\phi}_i(A^{*j}(\xi)) \\
&\quad \times \overline{\widehat{f}}(\xi) e^{-i\langle A^{*j}(\xi)|\gamma\rangle}\, d\xi \Big|^2 \\
&= \frac{1}{(2\pi)^{2n} d^j} \sum_i \sum_\gamma \Big| \int_{\mathbb{R}^n} \widehat{\phi}_i(\xi) \\
&\quad \times \overline{\widehat{f}}(A^{*-j}(\xi)) e^{-i\langle \gamma|\xi\rangle}\, d\xi \Big|^2 \\
&= \frac{1}{(2\pi)^n d^j} \sum_i \int_{[-\pi,\pi)^n} \Big| \sum_l \widehat{\phi}_i(\xi + 2\pi l) \\
&\quad \times \overline{\widehat{f}}(A^{*-j}(\xi + 2\pi l)) \Big|^2\, d\xi \\
&= \frac{1}{(2\pi)^n d^j} \sum_i \int_{[-\pi,\pi)^n} \Big| \sum_{|l| \ge l_j(\xi)} \widehat{\phi}_i(\xi + 2\pi l) \\
&\quad \times \overline{\widehat{f}}(A^{*-j}(\xi + 2\pi l)) \Big|^2\, d\xi \\
&\le \frac{1}{(2\pi)^n d^j} \sum_i \int_{[-\pi,\pi)^n} \sum_{|l| \ge l_j(\xi)} |\widehat{\phi}_i(\xi + 2\pi l)|^2 \\
&\quad \times \sum_{|l| \ge l_j(\xi)} |\widehat{f}(A^{*-j}(\xi + 2\pi l))|^2\, d\xi \\
&= \frac{1}{(2\pi)} \int_{[-\pi,\pi)^n} \left[\sum_i \sum_{|l| \ge l_j(\xi)} |\widehat{\phi}_i(\xi + 2\pi l)|^2 \right] \\
&\quad \times \left[\frac{1}{d^j} \sum_l |\widehat{f}(A^{*-j}(\xi + 2\pi l))|^2 \right] d\xi \\
&= \frac{1}{(2\pi)^n} \int_{[-\pi,\pi)^n} \left[\sum_i \sum_{|l| \ge l_j(\xi)} |\widehat{\phi}_i(\xi + 2\pi l)|^2 \right] \\
&\quad \times \left[\sum_k f * f^*(A^j(k)) e^{-i\langle k|\xi\rangle} \right] d\xi \\
&\le \frac{1}{(2\pi)^n} \int_{[-\pi,\pi)^n} \left[\sum_i \sum_{|l| \ge l_j(\xi)} |\widehat{\phi}_i(\xi + 2\pi l)|^2 \right] \\
&\quad \times \left[\sum_k |f * f^*(A^j(k))| \right] d\xi \\
&\le C_f \int_{[-\pi,\pi)^n} \sum_i \sum_{|l| \ge l_j(\xi)} |\widehat{\phi}_i(\xi + 2\pi l)|^2\, d\xi,
\end{aligned}
$$

which approaches 0 as j goes to infinity by the Dominated Convergence Theorem.

We have that the sequence $\|P_j(f)\|^2$ is nondecreasing, as j approaches ∞, for every $f \in L^2$, and is bounded above by $\|f\|^2$. Moreover, $\|f\|^2 = \lim_{j\to\infty} \|P_j(f)\|^2$ for every $f \in L^2$ if and only if $\cup V_j$ is dense in $L^2(\mathbb{R}^n)$. It will suffice to verify that this limit holds for every $f \in L^2(\mathbb{R}^n)$ whose Fourier transform $\widehat{f}$ has compact support. Because the functions $\{\gamma(\phi_i)\}$ form a normalized tight frame for V_0, we know that the functions $\{\delta^j(\gamma(\phi_i))\}$ form a normalized tight frame for V_j. Therefore, we have for $\widehat{f}$ of compact support and j sufficiently large,

$$\begin{aligned}
\|P_j(f)\|^2 &= \sum_i \sum_\gamma |\langle \delta^j(\gamma(\phi_i)) \mid f\rangle|^2 \\
&= \frac{1}{d^j(2\pi)^{2n}} \sum_i \sum_\gamma \Big| \int_{\mathbb{R}^n} e^{i\langle \gamma | A^{*-j}(\xi)\rangle} \widehat{\phi}_i(A^{*-j}(\xi)) \overline{\widehat{f}}(\xi)\, d\xi \Big|^2 \\
&= \frac{d^j}{(2\pi)^{2n}} \sum_i \sum_\gamma \Big| \int_{[-\pi,\pi)^n} e^{i\langle \gamma | \omega\rangle} \widehat{\phi}_i(\omega) \overline{\widehat{f}}(A^{*j}(\omega))\, d\omega \Big|^2 \\
&= \frac{d^j}{(2\pi)^{n}} \sum_i \int_{[-\pi,\pi)^n} |\widehat{\phi}_i(\omega) \overline{\widehat{f}}(A^{*j}(\omega))|^2\, d\omega \\
&= \frac{1}{(2\pi)^{n}} \sum_i \int_{A^{*j}([-\pi,\pi)^n)} |\widehat{\phi}_i(A^{*-j}(\xi)) \overline{\widehat{f}}(\xi)|^2\, d\xi \\
&= \frac{1}{(2\pi)^{n}} \int_{A^{*j}([-\pi,\pi)^n)} \sum_i |\widehat{\phi}_i(A^{*-j}(\xi))|^2 |\widehat{f}(\xi)|^2\, d\xi.
\end{aligned}$$

Since this sequence is bounded above by $\|f\|^2$ and (eventually) nondecreasing for all f's whose Fourier transforms have a common compact support, we deduce that the functions $\sum_i |\widehat{\phi}_i(A^{*-j}(\xi))|^2$ are almost everywhere eventually nondecreasing and bounded above by 1. Therefore, the limsup in (3) is actually always a limit. Finally, the preceding calculation shows that the sequence $\{\|P_j(f)\|^2\}$ converges to $\|f\|^2$ for all such f's if (3) holds.

Using this result, we will now show how to build Fourier transforms of generalized scaling functions from GCMFs having specified properties. This process generalizes the classical infinite product construction of Mallat, Meyer, and Daubechies. To simplify the proofs, we consider here only the case of a bounded multiplicity function.

DEFINITION. We call a GCMF *lower triangular* if $h_{i,j} = 0$ for $j > i$.

Suppose $m : [-\pi,\pi)^n \to \{0,1,2,\dots c\}$ is a bounded function satisfying the hypotheses of Proposition 3.1, and $S_i = \{\omega \in [-\pi,\pi)^n : m(\omega) \geq i\}$ as before. Suppose $\{h_{i,j}\}$, for $1 \leq i,j \leq c$, is a lower triangular GCMF relative to m, and write M^n for the matrix product

$$M^n(\xi) = \prod_{j=1}^n \frac{1}{\sqrt{d}} h(A^{*-j}(\xi)).$$

For the purposes of the following proofs, we introduce another matrix product $\widetilde{M}^n$, defined inductively as follows, where the E_i's are sets guaranteed by Proposition

3.1 and χ_{E_i} are their characteristic functions. Let

$$\widetilde{M}^0 = \begin{pmatrix} \chi_{E_1} & 0 & 0 & \cdots & 0 \\ \chi_{E_2} - \chi_E & \chi_E & 0 & \cdots & 0 \\ \chi_{E_3} - \chi_E & 0 & \chi_E & \cdots & 0 \\ \vdots & & & \ddots & \\ \chi_{E_c} - \chi_E & 0 & 0 & \cdots & \chi_E \end{pmatrix},$$

and $\widetilde{M}^n(\xi) = \frac{1}{\sqrt{d}} h(A^{*-1}\xi)\widetilde{M}^{n-1}(A^{*-1}\xi)$ for $n > 0$. We will need the following technical lemma.

LEMMA 3.3. *Let the notation be as in the preceding paragraph. Suppose that $m > 0$ on a neighborhood of the origin, and assume that the sequence $\{M^n(\xi)\}$ converges to a (lower triangular) matrix $M(\xi)$ for almost all ξ. Then:*

(1) *The sequence $\{\widetilde{M}^n(\xi)\}$ converges almost everywhere to the matrix product $\widetilde{M}(\xi) = M(\xi)K$, where K is the constant matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ \cdots & & & & & \\ -1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(2) *For all $1 \le i \le c$, we have*

$$M_{i,1}(\xi) = \sum_{j=1}^{c} \widetilde{M}_{i,j}(\xi).$$

(3) *For all i and i' we have*

$$\sum_{l\in\mathbb{Z}^n}\sum_{j=1}^{c} \widetilde{M}^n_{i,j}(\omega + 2\pi l)\overline{\sum_{j'=1}^{c} \widetilde{M}^n_{i',j'}(\omega + 2\pi l)} = \begin{cases} 0 & \text{for } i \ne i' \\ \chi_{S_i}(\omega) & \text{for } i = i' \end{cases}.$$

(4) *The entries $M_{i,1}$ belong to $L^2(\mathbb{R}^n)$.*

PROOF. By Proposition 3.1, E_1 contains a neighborhood of the origin, and thus, for almost all $\xi \in \mathbb{R}^n$, we must have that $A^{*-j}(\xi) \notin E_i$ for $i > 1$ and all large enough j. Parts (1) and (2) then follow from a straightforward matrix computation.

We now show part (3) by induction on n. If $n = 0$, we have

$$\begin{aligned} \sum_{l\in\mathbb{Z}^n}\sum_{j=1}^{c} \widetilde{M}^0_{i,j}(\omega + 2\pi l)\overline{\sum_{j'=1}^{c} \widetilde{M}^0_{i',j'}(\omega + 2\pi l)} &= \sum_{l\in\mathbb{Z}^n} \chi_{E_i}(\omega + 2\pi l)\chi_{E_{i'}}(\omega + 2\pi l) \\ &= \begin{cases} 0 & \text{if } i \ne i' \\ \chi_{S_i}(\omega) & \text{if } i = i' \end{cases} \end{aligned}$$

Now, assume

$$\sum_{l\in\mathbb{Z}^n}(\sum_j \widetilde{M}^n_{i,j}(\omega+2\pi l))\overline{(\sum_{j'}\widetilde{M}^n_{i',j'}(\omega+2\pi l))} = \begin{cases} 0 & \text{if } i\neq i' \\ \chi_{S_i}(\omega) & \text{if } i=i' \end{cases}.$$

Then

$$\begin{aligned}\sum_{l\in\mathbb{Z}^n}(\sum_j &\widetilde{M}^{n+1}_{i,j}(\omega+2\pi l))\overline{(\sum_{j'}\widetilde{M}^{n+1}_{i',j'}(\omega+2\pi l))} \\ &= \sum_{l\in\mathbb{Z}^n}\left(\sum_j\sum_k \frac{1}{\sqrt{d}}h_{i,k}(A^{*-1}(\omega+2\pi l))\widetilde{M}^n_{k,j}(A^{*-1}(\omega+2\pi l))\right) \\ &\qquad\times\left(\overline{\sum_{j'}\sum_{k'}\frac{1}{\sqrt{d}}h_{i',k'}(A^{*-1}(\omega+2\pi l))\widetilde{M}^n_{k',j'}(A^{*-1}(\omega+2\pi l))}\right).\end{aligned}$$

As described at the beginning of Section 2, we write $l = l_q + A^*p$, where l_q is a coset representative of $\mathbb{Z}^n/A^*\mathbb{Z}^n$ and p is an integer lattice point. Then $A^{*-1}(\omega+2\pi l) = A^{*-1}(\omega+2\pi l_q)+2\pi p$. By periodicity of h, our expression becomes

$$\begin{aligned}&\sum_{q=0}^{d-1}\sum_k\sum_{k'} h_{i,k}(A^{*-1}(\omega+2\pi l_q))\overline{h_{i',k'}(A^{*-1}(\omega+2\pi l_q))} \\ &\quad\times\sum_{p\in\mathbb{Z}^n}\sum_j\sum_{j'}\widetilde{M}^n_{k,j}(A^{*-1}(\omega+2\pi l_q)+2\pi p)\overline{\widetilde{M}^n_{k',j'}(A^{*-1}(\omega+2\pi l_q)+2\pi p)}.\end{aligned}$$

By the inductive assumption, the expression then becomes

$$\sum_q\sum_k(h_{i,k}(A^{*-1}(\omega+2\pi l_q))\overline{h_{i',k}(A^{*-1}(\omega+2\pi l_q))})\chi_{S_k}(A^{*-1}(\omega+2\pi l_q)).$$

Since $h_{i,k}$ is supported on S_k and h is a GCMF, we then have

$$\sum_q\sum_k h_{i,k}(A^{*-1}(\omega+2\pi l_q))\overline{h_{i',k}(A^{*-1}(\omega+2\pi l_q))} = \begin{cases} 0 & \text{if } i\neq i' \\ \chi_{S_i}(\omega) & \text{if } i=i' \end{cases}.$$

This finishes the proof of part (3).

Finally, we use part (3) to show that the components $M_{i,1}$ of the matrix product M belong to $L^2(\mathbb{R}^n)$. We integrate both sides of (3) over $[-\pi,\pi)^n$ to see that $\|\sum_j \widetilde{M}^n_{i,j}\| \le 1$ and then apply Fatou's Lemma and part (2).

THEOREM 3.4. *As in the preceding lemma, suppose $m > 0$ on a neighborhood of the origin, and that the sequence $\{M^n(\xi)\}$ converges to a (lower triangular) matrix $M(\xi)$ for almost all ξ. Suppose in addition that the entries $\widetilde{M}^n_{i,j}$ converge to $\widetilde{M}_{i,j}$ in $L^2(\mathbb{R}^n)$ and that the function $M_{1,1}$ is nonzero on a neighborhood of the origin. Then the functions $\{M_{i,1}\}$ are the Fourier transforms of a set $\{\phi_i\}$*

of generalized scaling functions for a GMRA $\{V_j\}$, whose associated multiplicity function coincides with the given function m.

PROOF. Define $\widehat{\phi_i}$ to equal $M_{i,1}$. By (4) of the preceding lemma, we know that the $\widehat{\phi_i}$'s are in $L^2(\mathbb{R}^n)$. We will show that the functions $\{\phi_i\}$ satisfy the three conditions of Theorem 3.2.

To see that the ϕ_i's satisfy (1) of Theorem 3.2, let $\widehat{\phi_i^n} = \sum_j \widetilde{M}_{i,j}^n$. Then, by part (3) of the preceding lemma,

$$\sum_{l\in\mathbb{Z}^n} \widehat{\phi_i^n}(\omega+2\pi l)\overline{\widehat{\phi_{i'}^n}(\omega+2\pi l)} = \begin{cases} 0 & \text{for } i \neq i' \\ \chi_{S_i}(\omega) & \text{for } i = i' \end{cases}.$$

The left hand side of this equation is a function on $[-\pi,\pi)^n$ whose γ^{th} Fourier coefficient is $\langle \gamma\phi_i^n \mid \phi_{i'}^n\rangle$. Thus, $\langle \gamma\phi_i^n \mid \phi_{i'}^n\rangle$ is the γ^{th} Fourier coefficient of the right hand side of this equation as well. Since (using part (2) of the preceding lemma) we are assuming that $\phi_i^n \to \phi_i$ in $L^2(\mathbb{R}^n)$, we have $\langle \gamma\phi_i^n \mid \phi_{i'}^n\rangle \to \langle \gamma\phi_i \mid \phi_{i'}\rangle$, so that $\langle \gamma\phi_i \mid \phi_{i'}\rangle$ must also be the γ^{th} Fourier coefficient of the function which is 0 when $i \neq i'$ and χ_{S_i} when $i = i'$. But $\langle \gamma\phi_i \mid \phi_{i'}\rangle$ is the γ^{th} Fourier coefficient for the function $\sum_{l\in\mathbb{Z}} \widehat{\phi_i}(\omega+2\pi l)\overline{\widehat{\phi_{i'}}(\omega+2\pi l)}$. Equation (1) of Theorem 3.2 follows.

Next we show that the collection $\{\widehat{\phi_i}\}$ satisfies property (2) of Theorem 3.2. Thus,

$$\begin{aligned}
\widehat{\phi_i}(A^*\xi) &= M_{i,1}(A^*\xi) \\
&= \left[\prod_{j=1}^{\infty} \frac{1}{\sqrt{d}} h((A^*)^{1-j}\xi)\right]_{i,1} \\
&= \left[\prod_{j=0}^{\infty} \frac{1}{\sqrt{d}} h((A^*)^{-j}\xi)\right]_{i,1} \\
&= \sum_k \frac{1}{\sqrt{d}} h_{i,k}(\xi) \left[\prod_{j=1}^{\infty} \frac{1}{\sqrt{d}} h((A^*)^{-j}\xi)\right]_{k,1} \\
&= \sum_k \frac{1}{\sqrt{d}} h_{i,k}(\xi)\widehat{\phi_k}(\xi).
\end{aligned}$$

We will finish the proof by showing that $\{\phi_i\}$ satisfy property (3) of Theorem 3.2. By iterating property (2) for the $\{\widehat{\phi_i}\}$, we have

$$\widehat{\phi_1}(\xi) = \lim_{j\to\infty} \left(\prod_{k=1}^{j} \frac{1}{\sqrt{d}} h_{1,1}(A^{*-k}\xi)\right) \widehat{\phi_1}(A^{*-j}\xi).$$

Because the GCMF $\{h_{i,j}\}$ is lower triangular, this infinite product converges to $\widehat{\phi_1}(\xi)$, implying that $\lim_{j\to\infty} \widehat{\phi_1}(A^{*-j}(\xi))$ must exist and equal 1 wherever $\widehat{\phi_1}(\xi) \neq 0$, which by hypothesis includes a neighborhood of the origin. Since this limit is invariant under A^*, and A^* is expansive, we have $\lim_{j\to\infty} \widehat{\phi_1}(A^{*-j}(\xi)) = 1$ a.e., which clearly implies condition (3) of Theorem 3.2.

The preceding theorem shows how to construct generalized scaling functions for a GMRA from filters having special properties. The next corollary completes our generalization of [Mal1], [Mey] and [Dau], by describing how to construct the related wavelet basis from these generalized scaling functions and a complementary filter.

COROLLARY 3.5. *Let m, $\{h_{i,j}\}$, and $\{\phi_i\}$ be as in the preceding theorem. Let $\{g_{k,j}\}$ be any complementary conjugate mirror filter associated to the GCMF $\{h_{i,j}\}$. For each k, define a function ψ_k by*

$$\sqrt{d}\widehat{\psi_k}(A^*(\xi)) = \sum_j g_{k,j}(\xi)\widehat{\phi_j}(\xi).$$

Then the collection $\{\psi_k\}$ is a frame multiwavelet for $L^2(\mathbb{R}^n)$. If m satisfies the consistency equation $m(\omega) + N = \sum m(\omega_l)$, then $\psi_1, \dots, \psi_N$ form a multiwavelet for $L^2(\mathbb{R}^n)$.

PROOF. Part (2) of Theorem 2.5 gives a formula for the wavelet basis in terms of the complementary filter and the unitary map $J : V_0 \to \bigoplus L^2(S_j)$, that is

$$\delta^{-1}(\psi_k) = J^{-1}(\bigoplus_j g_{k,j}).$$

In the present case, such a map J is defined in the proof of Theorem 3.2 and is given by

$$[J(f)]_j(\omega) = \sum_\gamma \langle f \mid \gamma(\phi_j)\rangle e^{i\langle\gamma|\omega\rangle}\chi_{S_j}(\omega).$$

We see that the inverse of this J is given by

$$J^{-1}(\bigoplus_j v_j) = \sum_j \sum_{\gamma\in\mathbb{Z}^n} c_{j,\gamma}\gamma(\phi_j),$$

where the $c_{j,\gamma}$'s are the Fourier coefficients of the function v_j. We complete the argument by computing the Fourier transforms of both sides of the equation above defining ψ_k. The Fourier transform of $\delta^{-1}(\psi_k)$ is given by $\sqrt{d}\widehat{\psi_k}(A^*(\xi))$, while the Fourier transform of $J^{-1}(\bigoplus_j g_{k,j})$ is given by

$$\sum_j \sum_\gamma c_{j,\gamma}e^{i\langle\gamma|\xi\rangle}\widehat{\phi_j}(\xi) = \sum_j g_{k,j}(\xi)\widehat{\phi_j}(\xi).$$

The above constructions rely on a judicious choice of a GCMF. In the next theorem we show how to construct GCMFs to which Theorem 3.4 applies. To do this, we will need to impose some restrictions on the multiplicity function m. However, these restrictions are met by almost all of the multiplicity functions that have received attention in the literature.

THEOREM 3.6. *Let* $m : [-\pi, \pi)^n \mapsto \{0, 1, 2, \cdots c\}$ *be a bounded function that satisfies the conditions of Proposition 3.1. In addition, suppose* $m(\omega) > 0$ *on a neighborhood of the origin, and* $\widetilde{m}(\omega) \geq c - 1$*. Then a GCMF that satisfies the conditions of Theorem 3.4 can be constructed explicitly from* m*.*

PROOF. We begin by taking $h_{1,1} = \sqrt{d}\chi_{A^{*-1}E_1}$, where E_1 is defined as in Proposition 3.1, and $h_{1,j} = 0$ for $j > 1$. Note that $h_1 = \oplus h_{1,j}$ satisfies

$$\sum_j \sum_{l=0}^{d-1} |h_{1,j}(\omega_l)|^2 = d\chi_{S_1}(\omega),$$

as in the definition of a generalized conjugate mirror filter, for exactly one of the preimages, ω_l, of ω is in $A^{*-1}E_1$ mod 2π if and only if $\omega \in S_1$. Since by hypothesis S_1 contains a neighborhood of the origin, Proposition 3.1 ensures that E_1 does as well. Thus this definition of h_1 will guarantee that $M_{1,1}(\omega) \neq 0$ on a neighborhood of 0. To construct the remainder of the $h_{i,j}$ we use the following procedure: As in the proof of Theorem 2.5, partition $[-\pi, \pi)^n$ into a countable collection of sets of the form

$$P_{l_0, l_1, \cdots l_{d-1}, \widetilde{l}} = \{\omega \in [-\pi, \pi)^n : m(\omega_r) = l_r, \widetilde{m}(\omega) = \widetilde{l}\}.$$

We will define the functions $h_{i,j}$, $i \geq 2$, piecewise on the sets $P_{l_0, l_1, \cdots l_{d-1}, \widetilde{l}}$ by fixing an ω, and building the vectors $\vec{h}_i^\omega = (h_{i,1}(\omega_0), \cdots, h_{i,l_0}(\omega_0), h_{i,1}(\omega_1), \cdots, h_{i,l_1}(\omega_1), \cdots, h_{i,1}(\omega_{d-1}), \cdots, h_{i,l_{d-1}}(\omega_{d-1}))$ as in Lemma 2.4. Once we have built vectors $\vec{h}_i^\omega \in \mathbb{C}^{m(\omega)+\widetilde{m}(\omega)}$ for $2 \leq i \leq m(\omega)$, for almost all $\omega \in [-\pi, \pi)^n$, we will have determined the functions $h_{i,j}$ a.e.. In particular, knowledge of $h_{i,j}$ in the ω_0 positions will determine the function $h_{i,j}$ on a neighborhood of the origin.

In order to have the resulting functions $h_{i,j}$ be a GCMF, we must have the vectors $\vec{h}_i^\omega$ satisfy the first property of Lemma 2.4. Thus, for almost all $\omega \in P_{l_0, l_1, \cdots l_{d-1}, \widetilde{l}}$, we must have $m(\omega) = \sum_{r=0}^{d-1} l_r - \widetilde{l}$ vectors $\vec{h}_i^\omega$, $1 \leq i \leq m(\omega)$, which are pairwise orthogonal and each of norm $\sqrt{d}$. Our choice of h_1 gives us the first of these vectors, $\vec{h}_1^\omega$. Since our vectors are $m(\omega) + \widetilde{m}(\omega)$ long, and $m(\omega) + \widetilde{m}(\omega) \geq m(\omega) + c - 1$, we know that it is possible to construct enough pairwise orthogonal vectors. In order to meet the required conditions of the Theorem, our strategy will be to simplify the matrix product in Theorem 3.4 by taking each of our vectors to have a single nonzero entry of $\sqrt{d}$, each in a different position. (Note that $\vec{h}_1^\omega$ already has this property.) This will leave $\widetilde{m}(\omega) \geq c - 1$ extra positions we do not need to use for a nonzero entry in one of the vectors. In order to further simplify the matrix product, we would like $h_{i,j}(A^{*-k}(\omega)) = 0$ for $j \neq 1$ and k sufficiently large. This would have the effect of making the matrix product M have nonzero entries only in the first column. We will accomplish this by taking $h_{i,j}(\omega_0) = 0$ whenever $j \neq 0$. This will still allow us enough places to construct $m(\omega)$ orthogonal vectors, since we will be requiring 0's in $m(\omega_0) - 1 \leq c - 1$ positions, and we have seen that we have $c - 1$ positions we do not need. With these requirements and the need for the GCMF to be lower triangular in mind, we build the vectors $\vec{h}_i^\omega$ for $i \geq 2$ as follows: Let q_1 be the number of distinct preimages ω_r of ω under α (i.e. under A^* mod 2π) that are not in $A^{*-1}E_1$ and that satisfy $m(\omega_r) \geq 1$. For $t > 1$, let q_t be the number of distinct preimages $\omega_r \neq \omega_0$ of ω under α that satisfy $m(\omega_r) \geq t$. Then $d-1 \geq q_1 \geq q_2 \geq \cdots \geq q_c$, and $\sum_{t=1}^{c} q_t \leq \sum_{j=0}^{d-1} l_j = m(\omega) + \widetilde{m}(\omega)$. There are q_1 different points $\omega_r \notin A^{*-1}E_1$ that are in S_1 and thus q_1 positions of the form $h_{i,1}(\omega_r)$, $\omega_r \notin \text{support}(h_{1,1})$, in the vectors $\vec{h}_i^{\omega}$. We build the next q_1 vectors $\vec{h}_i^{\omega}$ by letting each take on the value $\sqrt{d}$ in the $j=1$ position at a different one of these points. Similarly, there are q_2 different positions of the form $h_{i,2}(\omega_r)$, $r \neq 0$ in the vectors $\vec{h}_i^{\omega}$; we build the next q_2 vectors $\vec{h}_i^{\omega}$ by letting each take on the value $\sqrt{d}$ in the $j=2$ position at a different one of these points, etc. We have constructed $m(\omega)$ different vectors $\vec{h}_i^{\omega}$ that are orthogonal and have norm $\sqrt{d}$. By Lemma 2.4, these vectors determine functions $h_{i,j}$ that satisfy the definition of a GCMF. We have $h_{i,j}(\omega_r) = 0$ if $j > i$ since we built the vectors $\vec{h}_i^{\omega}$ in order of increasing i, with at least one with its nonzero element in the first column, then at least one with its nonzero element in the second column, etc.. Thus we have built a lower triangular GCMF. As usual, we extend the functions $h_{i,j}$ periodically.

It remains to show that the components of $\widetilde{M^n}(\xi)$ converge to those of $\widetilde{M}(\xi)$ in $L^2(\mathbb{R}^n)$. We have built the $h_{i,j}$ such that $h_{i,j}(\omega_0) = 0$ for $j > 1$, so that $h_{i,j}(\omega) = 0$ on a neighborhood of the origin for $j > 1$. As a consequence, $\lim_{k\to\infty} h_{i,j}(A^{*-k}(\omega)) = 0$ if $j > 1$. Thus, we see that both $\widetilde{M}$ and $\widetilde{M^n}$ for large n can have nonzero entries only in the first column. These potentially nonzero entries are of the form $\widetilde{M}_{i,1}(\omega) = f_i(\omega)\prod_{j=j_0}^{\infty} \frac{1}{\sqrt{d}} h_{1,1}(A^{*-j}\omega)$ and $\widetilde{M}_{i,1}^n(\omega) = \chi_{E_1}(A^{*-n}\omega) f_i(\omega) \prod_{j=j_0}^{n} \frac{1}{\sqrt{d}} h_{1,1}(A^{*-j}\omega)$, where the functions f_i hold the contribution of the first $j_0 - 1$ matrix factors that are not 0 outside the first column. Since for j sufficiently large, $h_{1,1}(A^{*-j}\omega) = \sqrt{d}$, the product converges a.e., so by Lemma 3.3, $\widetilde{M}_{i,1} = M_{i,1}$ is in $L^2(\mathbb{R}^n)$. We will use $M_{i,1}$ to dominate the $\widetilde{M}_{i,1}^n$ and thus show that the components of $\widetilde{M}^n$ converge in $L^2(\mathbb{R}^n)$. We have that $\widetilde{M}_{i,1}^n(\omega) = \begin{cases} \frac{M_{i,1}(\omega)}{\prod_{j=n+1}^{\infty} \frac{1}{\sqrt{d}} h_{1,1}(A^{*-j}\omega)} & \text{if} \quad \omega \in A^{*n}E \\ 0 & \text{if} \quad \omega \notin A^{*n}E \end{cases}$, which concludes the proof, since by our definition of $h_{1,1} = \chi_{A^{*-1}E_1}$ and the fact that $E_1 \subset A^*E_1$, we have that $\prod_{j=n+1}^{\infty} \frac{1}{\sqrt{d}} h_{1,1}(A^{*-j}\omega) = 1$ on $A^{*n}E$.

The canonical GCMFs constructed in Theorem 3.6 result in generalized scaling functions whose Fourier transforms are characteristic functions, and thus produce wavelet set (frame) multi-wavelets. To meet our goal of building wavelets between the known wavelet set and MRA examples, we will alter this canonical GCMF using classical conjugate mirror filters such as those employed by Daubechies. In general, there are many ways to accomplish this. In the following corollary, we give one possible construction (or algorithm) for the case in which $m(\omega) \leq 2$ almost everywhere. The condition $m(\omega) \leq 2$ ensures that the restriction on $\widetilde{m}$ in Theorem 3.6 is always met whenever m has the potential to be associated with an (orthogonal) wavelet or multiwavelet. Recall that we write α for the homomorphism on $[-\pi,\pi)^n$ that sends ω to $A^*\omega$ mod 2π.

COROLLARY 3.7. *Let m be a multiplicity function (i.e. a function on $[-\pi,\pi)^n$ satisfying the conditions of Proposition 3.1) with $m(\omega) \leq 2$ a.e., $\widetilde{m}(\omega) \geq 1$ a.e., and $m(\omega) > 0$ on a neighborhood of the origin. Let $\{h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2}\}$ be the canonical GCMF constructed in Theorem 3.6. For each $\omega \in \text{support}(h_{1,1})$, choose* *(measurably in ω) any $z_\omega \in \ker(\alpha)$ such that $\omega + z_\omega \in S_1$. Let p be any measurable complex valued function defined on $[-\pi,\pi)^n$ (and extended periodically to $\mathbb{R}^n$) satisfying*

(1) *For $\omega \in \operatorname{support}(h_{1,1})$, $|p(\omega)|^2 + |p(\omega + z_\omega)|^2 = d$ if $z_\omega \neq 0$ and $|p(\omega)|^2 = d$ if $z_\omega = 0$.*

(2) *p is differentiable at $\omega = 0$ and $|p(0)| = \sqrt{d}$.*

(3) *$\exists$ a measurable set $B \subset E_1$ (possibly empty), a nonnegative integer J, and a positive constant λ such that $|p(A^{*-j}\omega)| > \lambda$ for all $\omega \in E_1 \setminus B$ and all $j \geq 1$, and for each $\omega \in B$ there exists a j between 0 and J for which $p(A^{*j}(\omega)) = 0$.*

Define

$$h'_{1,1}(\omega) = \begin{cases} p(\omega) & \text{if } \omega \in \text{ support } (h_{1,1}) \cup \{\nu + z_\nu : \nu \in \text{ support } (h_{1,1})\} \\ 0 & \text{otherwise} \end{cases},$$

$$h'_{2,1}(\omega) = \begin{cases} -\overline{p(\omega + z_\omega)} & \text{if } \omega \in \text{support } (h_{1,1}), z_\omega \neq 0, A^*(\omega) \in S_2 \\ \overline{p(\nu)} & \text{if } \omega = \nu + z_\nu, \nu \in \text{support } (h_{1,1}), z_\nu \neq 0, A^*(\omega) \in S_2 \\ h_{2,1}(\omega) & \text{otherwise} \end{cases},$$

and $h'_{1,2} = h'_{2,2} = 0$. Then, $\{h'_{1,1}, h'_{1,2}, h'_{2,1}, h'_{2,2}\}$ form a GCMF that satisfies the hypotheses of Theorem 3.4.

PROOF. Note that, by the consistency equation, the q_1 of the proof of Theorem 3.6 satisfies $q_1 \geq 1$ if $\omega \in S_2$. Thus, we have the canonical GCMF constructed there satisfying $h_{1,2} = h_{2,2} = 0$. Recall also that for each $\omega \in S_1$, there is a unique preimage of ω under α, ω_{l_0} which is in $\operatorname{support}(h_{1,1}) = A^{*-1}E_1 \pmod{2\pi}$, and note that $\omega_{l_0} + z_{\omega_{l_0}}$ is also one of the preimages. With these observations, a routine calculation shows that $\{h'_{1,1}, h'_{1,2}, h'_{2,1}, h'_{2,2}\}$ satisfies the mirror equation

$$\sum_j \sum_{l=0}^{d-1} h'_{i,j}(\omega_l)\overline{h'_{k,j}(\omega_l)} = \begin{cases} d\chi_{S_i}(\omega) & i = k \\ 0 i \neq k \end{cases}.$$

Since $h'_{1,2} = h'_{2,2} = 0$, we have that the components of the matrix product $M^n(\xi) = \prod_{j=1}^n \frac{1}{\sqrt{d}} h'(A^{*-j}(\xi))$ are of the form:

$$M^n_{1,1}(\xi) = \prod_{j=1}^n \frac{1}{\sqrt{d}} h'_{1,1}(A^{*-j}\xi)$$

$$M^n_{2,1}(\xi) = \frac{1}{\sqrt{d}} h'_{2,1}(A^{*-1}\xi) \prod_{j=2}^n \frac{1}{\sqrt{d}} h'_{1,1}(A^{*-j}\xi)$$

$$M^n_{1,2}(\xi) = M^n_{2,2}(\xi) = 0.$$

Thus, a.e. convergence of these components follows immediately from condition (2). Condition (2) also ensures that $M_{1,1}$, the limit of the $\widetilde{M}^n_{1,1}(\xi)$, is nonzero on a neighborhood of the origin.

It remains to show that $\widetilde{M}^n_{i,1} = \chi_{E_1}(A^{*-n}\xi)\frac{1}{\sqrt{d}}h_{i,1}(A^{*-1}\xi)\prod_{j=2}^n \frac{1}{\sqrt{d}}h_{1,1}(A^{*-j}\xi)$ converge in L^2. Since we already know that $\widetilde{M}^n_{i,1}$ converge a.e., we will use the Dominated Convergence Theorem. If $A^{*-n}\xi \notin E_1$,we have that $\widetilde{M}^n_{i,1}(\xi) = 0$. Thus, it will suffice to show that $\widetilde{M}^n_{i,1}(\xi)$ are all bounded by the same dominating function on $A^{*n}E_1$. For $N \geq J$ and $\xi \in A^{*n}B$, condition (3) ensures that $\widetilde{M}^n_{i,1}(\xi) = 0$. If $\xi \in A^{*n}E_1 \setminus B$, we have

$$\widetilde{M}^n_{i,1}(\xi) = \frac{M_{i,1}(\xi)}{M_{1,1}(A^{*-n}\xi)}.$$

Condition (3) together with condition (2) implies that there exists a constant $\lambda' > 0$ (independent of n) such that $M_{1,1}(A^{*-n}\xi) > \lambda'$ for $\xi \in A^{*n}(E_1 \setminus B)$. Thus we can use $\frac{M_{i,1}(\xi)}{\lambda'}$ (which is in $L^2(\mathbb{R}^n)$ by Lemma 3.3) as a dominating function.

4. Examples

In this Section we will use the techniques of Sections 2 and 3 to build GCMFs then GMRAs and finally wavelets for a variety of multiplicity functions. In our first example, we build a 2-wavelet for dilation by 2 in $L^2(\mathbb{R}^2)$ using the canonical GCMF from Theorem 3.6. It is well-known that any multiwavelet in 2 dimensions built from an MRA must have exactly 3 elements. Of course the multiwavelet we build here is associated to a GMRA that is not an MRA, so the fact that it has only 2 elements in it is not a contradiction, but just a verification that GMRAs give rise to unexpected wavelet phenomena. Unlike the known single wavelets in 2 dimensions, which all involve fractal-like sets, the 2-wavelet constructed here is quite simple.

EXAMPLE 4.1. Define $A = [-\frac{2\pi}{3}, \frac{2\pi}{3})$ and $B = [\frac{2\pi}{3}, \pi) \cup [-\pi, \frac{-2\pi}{3})$. Let S_2 be the subset of $\mathbb{R}^2$ consisting of the four squares given by $B \times B$. Let S_1 be $S_2 \cup A \times A$, and define a multiplicity function by $m = \chi_{S_1} + \chi_{S_2}$. It is easy to verify that m satisfies the conditions of Proposition 3.1 for $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, with $\widetilde{m}(\omega) = \sum m(\omega_l) - m(\omega) \equiv 2$. Thus, m is the multiplicity function associated to the GMRA determined by a 2-wavelet for dilation by 2 in $L^2(\mathbb{R}^2)$.

By Proposition 3.1, here $E_1 = S_1$, so the canonical $h_{1,1} = 2\chi_{\frac{1}{2}S_1}$. As always, we take $h_{1,2} = 0$. Thus, for $\omega \in A \times A$, the vector $\vec{h}^\omega_1$ is given by $\vec{h}^\omega_1 = \langle h_{1,1}(\frac{\omega}{2}), h_{1,1}(\frac{\omega}{2} + (\pi,\pi)), h_{1,2}(\frac{\omega}{2} + (\pi,\pi))\rangle = \langle 2,0,0\rangle$, while for $\omega \in S_2$, $\vec{h}^\omega_1 = \langle h_{1,1}(\frac{\omega}{2}), h_{1,1}(\frac{\omega}{2} + (0,\pi)), h_{1,1}(\frac{\omega}{2} + (\pi,0)), h_{1,1}(\frac{\omega}{2} + (\pi,\pi))\rangle = \langle 2,0,0,0\rangle$. The vector $\vec{h}^\omega_2$ is defined only for $\omega \in S_2$, and must be orthogonal to $\vec{h}^\omega_1$ there, so we can take $\vec{h}^\omega_2 = \langle 0,0,0,2\rangle$, resulting in the function $h_{2,1}$ being defined by $h_{2,1} = 2\chi_{\left([\frac{-2\pi}{3},\frac{-\pi}{2})\cup[\frac{\pi}{2},\frac{2\pi}{3})\right)\times\left([\frac{-2\pi}{3},\frac{-\pi}{2})\cup[\frac{\pi}{2},\frac{2\pi}{3})\right)}$, and $h_{2,2} = 0$. Following Theorem 3.4, we can now use this canonical GCMF to build the generalized scaling functions:

$$\widehat{\phi}_1(\xi) = \prod_{j=1}^{\infty} \frac{1}{2}h_{1,1}(\frac{\xi}{2^j}) = \chi_{S_1}$$

and

$$\widehat{\phi}_2(\xi) = \frac{1}{2}h_{2,1}(\frac{\xi}{2})\widehat{\phi}_1(\frac{\xi}{2}) = \chi_{\left([-\frac{4\pi}{3},-\pi)\cup[\pi,\frac{4\pi}{3})\right)\times\left([\frac{-4\pi}{3},-\pi)\cup[\pi,\frac{4\pi}{3})\right)}.$$

Next, we use Theorem 2.4 to build an associated CCMF $\{g_{i,j}\}$. Since $\widetilde{m} \equiv 2$, we need two vectors, $\vec{g}_1^{\omega}$ and $\vec{g}_2^{\omega}$ which are orthogonal to our $\vec{h}_i^{\omega}$ vectors. Recalling the form of the vectors $\vec{h}_i^{\omega}$, we see that on $A \times A$, we can take $\vec{g}_1^{\omega} = \langle 0, 2, 0\rangle$ and $\vec{g}_2^{\omega} = \langle 0, 0, 2\rangle$; on S_2 we can take $\vec{g}_1^{\omega} = \langle 0, 2, 0, 0\rangle$ and $\vec{g}_2^{\omega} = \langle 0, 0, 2, 0\rangle$; and on the rest of $[-\pi, \pi)^2$ we can take $\vec{g}_1^{\omega} = \langle 2, 0\rangle$ and $\vec{g}_1^{\omega} = \langle 0, 2\rangle$. This results in the following CCMF:

$$g_{1,1} = 2\chi_{S_2 \cup \left(\frac{1}{2}B \times \left([\frac{-2\pi}{3}, \frac{-\pi}{2}) \cup [\frac{\pi}{2}, \frac{2\pi}{3})\right)\right) \cup \left(\frac{1}{2}A \times \frac{1}{2}B\right) \cup \left(\frac{1}{2}B \times \frac{1}{2}A\right)}$$

$$g_{1,2} = 0$$

$$g_{2,1} = 2\chi_{\left(\left([\frac{-2\pi}{3}, \frac{-\pi}{2}) \cup [\frac{\pi}{2}, \frac{2\pi}{3})\right) \times \frac{1}{2}B\right) \cup \left(\frac{1}{2}A \times \left([\frac{-2\pi}{3}, \frac{-\pi}{2}) \cup [\frac{\pi}{2}, \frac{2\pi}{3})\right)\right) \cup \left(\left([\frac{-2\pi}{3}, \frac{-\pi}{2}) \cup [\frac{\pi}{2}, \frac{2\pi}{3})\right) \times \frac{1}{2}A\right)}$$

$$g_{2,2} = 2\chi_{S_2}$$

Finally, using Corollary 3.5 we will build a 2-wavelet for dilation by 2 in $\mathbb{R}^2$. We have

$$\widehat{\psi}_1(\xi) = \frac{1}{2} g_{1,1}(\frac{\xi}{2})\widehat{\phi}_1(\frac{\xi}{2}),$$

and

$$\widehat{\psi}_2(\xi) = \frac{1}{2}\left(g_{2,1}(\frac{\xi}{2})\widehat{\phi}_1(\frac{\xi}{2}) + g_{2,2}(\frac{\xi}{2})\widehat{\phi}_2(\frac{\xi}{2})\right).$$

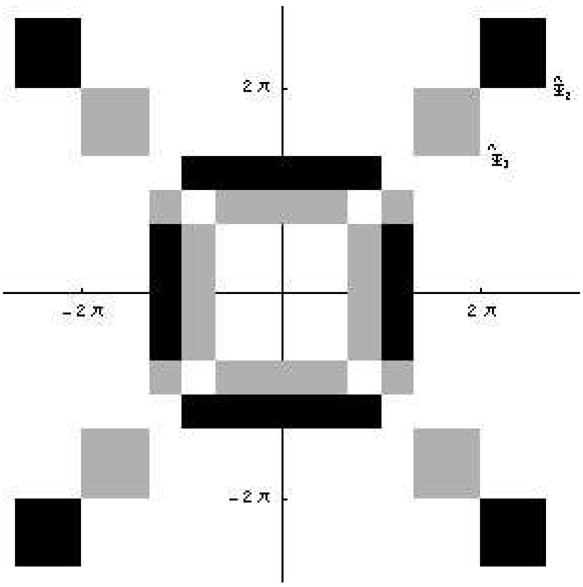
If we let the set $C = [\frac{-4\pi}{3}, -\pi) \cup [\pi, \frac{4\pi}{3})$, then we can write $\widehat{\psi_1}$ and $\widehat{\psi_2}$ as

$$\widehat{\psi}_1 = \chi_{2S_2 \cup ((A \cup C) \times B) \cup (B \times A)},$$

and

$$\widehat{\psi}_2 = \chi_{(2C \times 2C) \cup ([-\pi, \pi) \times C) \cup (C \times A)}.$$

See the picture below.



EXAMPLE 4.2. We give next an example of an FMRA wavelet, constructed via Theorem 3.6 and Corollary 3.7. This example will be relative to dilation by 2 in $L^2(\mathbb{R})$, although totally analogous constructions work in higher dimensions and for more general dilation matrices. For other examples of FMRA's in the literature, see [BL], [PSWX], [PSW], and [Han]. Recall that an FMRA is a GMRA for which there is a single generalized scaling function $\phi \in V_0$. The multiplicity function associated to an FMRA is the characteristic function χ_S of some subset of $\widehat{\mathbb{Z}} \equiv [-\pi, \pi)$. (See the end of Section 2.) In order that $m = \chi_S$ satisfy the necessary consistency inequality, we see that for each $\omega \in S$, we must have at least one of the two preimages $\frac{\omega}{2}$ or $\frac{\omega}{2} + \pi$ is in S. In addition, our construction techniques require that m be nonzero on a neighborhood of the origin, i.e., that S contains a neighborhood of 0. Finally, to ensure that $m = \chi_S$ satisfies the necessary conditions in the hypotheses to Theorem 3.1 to be a multiplicity function, we will assume that, as a subset of $\mathbb{R}$, $S \subseteq 2S$.

To make the construction interesting, it is necessary to ensure that Corollary 3.7 can be used in a nontrivial way, i.e., that the support of $h \equiv h_{1,1}$ contains some points ω for which nonzero elements z_ω exist.

Consider the multiplicity function m given by the characteristic function of the set

$$S = [-\pi, \frac{-6\pi}{7}) \cup [\frac{-4\pi}{7}, \frac{4\pi}{7}) \cup [\frac{6\pi}{7}, \pi).$$

This set will show up again in the next example, where it will be the support of the Journé multiplicity function. Our aim here is to use Corollary 3.7 to construct a generalized scaling function ϕ and an associated frame wavelet ψ whose Fourier transforms are as smooth as possible. It is not clear to us whether or not these functions can be smooth everywhere in $\mathbb{R}$. What we can show is that, given any bounded interval (a, b), there exists a generalized scaling function ϕ and a frame wavelet ψ associated to this multiplicity function m both of whose Fourier transforms are C^∞ on the interval (a, b).

The canonical GCMF h associated to m described in Theorem 3.6 is the 2π periodization of the function $h = \sqrt{2}\chi_{\frac{1}{2}S}$. Let the kernel element z_ω as in Corollary 3.7 be π if $\omega \in [-\frac{\pi}{7}, \frac{\pi}{7}) \cup \pm[\frac{3\pi}{7}, \frac{\pi}{2})$ and 0 otherwise. Following the ideas in Corollary 3.7, we let p be a periodic real-valued C^∞ function defined on $[-\pi, \pi)$ that satisfies the MRA mirror equation $|p(\omega)|^2 + |p(\omega + \pi)|^2 = 2$, and $p(0) = \sqrt{2}$. As in Corollary 3.7, we define a new GCMF as follows:

$$h'(\omega) = \begin{cases} p(\omega) & \omega \in [-\frac{\pi}{7}, \frac{\pi}{7}) \cup \pm[\frac{3\pi}{7}, \frac{4\pi}{7}) \cup \pm[\frac{6\pi}{7}, \pi) \\ \sqrt{2} & \omega \in \pm[\frac{\pi}{7}, \frac{2\pi}{7}) \\ 0 & \text{otherwise} \end{cases}.$$

To ensure that h' is as continuous as possible, we impose the following additional conditions on the function p.

(1) $p(\pm\frac{\pi}{7}) = \sqrt{2}$, and consequently $p(\pm\frac{6\pi}{7}) = 0$.
(2) $p(\pm\frac{3\pi}{7}) = 0$, and $p(\pm\frac{4\pi}{7}) = \sqrt{2}$.

Observe that h' is C^∞ everywhere except for jump discontinuities (with all derivatives zero in a neighborhood of the discontinuity) at points congruent mod 2π to $\pm\frac{2\pi}{7}$ and $\pm\frac{4\pi}{7}$.

Recalling that the Fourier transform $\widehat{\phi}$ of the generalized scaling function will be defined by

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{2}} h'(\frac{\xi}{2^j}),$$

we see that the only potential discontinuities of $\widehat{\phi}$ will occur at points where one of these factors has a jump discontinuity, i.e., at points that are congruent to $\pm\frac{2\pi}{7}$, $\pm\frac{4\pi}{7}$ or $\pm\frac{8\pi}{7}$ mod 2π.

We wish now to impose extra conditions on the function p (equivalently on h') so that the Fourier transforms $\widehat{\phi}$ and $\widehat{\psi}$ will be as smooth as possible, i.e., C^∞ on a prescribed bounded interval. Thus, fix a positive integer K. Assume the following extra conditions on p.

(1) $p(\pi \pm \frac{\pi}{7\times 2^k}) = h'(\pi \pm \frac{\pi}{7\times 2^k}) = 0$ for all $0 \le k \le K$.

(2) $p(\pm\frac{\pi}{7\times 2^{K+1}}) = h'(\pm\frac{\pi}{7\times 2^{K+1}}) = 0$.

Since we already have $h'(\pm\frac{5\pi}{7}) = h'(\pm\frac{3\pi}{7}) = 0$, condition (1) actually holds for $-2 \le k \le K$. As we have noted, $\widehat{\phi}$ is not C^∞ (or even continuous) at a point ξ only if for some $j \ge 1$, $\frac{\xi}{2^j}$ is congruent to $\pm\frac{2\pi}{7}$ or $\pm\frac{4\pi}{7}$ mod 2π. However, $\widehat{\phi}$ would be C^∞ at such a point ξ if there exists a $k > j$ such that h' is C^∞ and equal to 0 at $\frac{\xi}{2^k}$, since

$$\begin{aligned}\widehat{\phi}(\xi) &= \prod_{l=1}^{j} \frac{1}{\sqrt{2}} h'(\frac{\xi}{2^l})\widehat{\phi}(\frac{\xi}{2^j}) \\ &= \prod_{l=1}^{k} \frac{1}{\sqrt{2}} h'(\frac{\xi}{2^l})\widehat{\phi}(\frac{\xi}{2^k}).\end{aligned}$$

Now, for any $\xi' = \frac{\xi}{2^j}$ of the form $\xi' = 2n\pi \pm \frac{2\pi}{7}$, with $0 < |n| < 2^K$, we write n uniquely as $2^k \times l$ where l is odd. Then $\widehat{\phi}(\xi')$ has a factor of $h'(\pi \pm \frac{\pi}{7\times 2^k})$, which is 0, showing that $\widehat{\phi}$ is C^∞ at ξ. The same argument shows that $\widehat{\phi}$ is C^∞ at ξ for ξ' of the form $2n\pi \pm \frac{4\pi}{7}$ or $2n\pi \pm \frac{8\pi}{7}$, for $0 < |n| < 2^K$. Finally, if ξ is of the form $\pm\frac{2\pi}{7}$, $\pm\frac{4\pi}{7}$, or $\pm\frac{8\pi}{7}$ (i.e., $n = 0$), then $\widehat{\phi}(\xi)$ contains the factor $h'(\frac{\pi}{7\times 2^{K+1}})$, making $\widehat{\phi}$ C^∞ at ξ. Therefore, $\widehat{\phi}$ is C^∞ on the interval $(-2^{K+1}\pi, 2^{K+1}\pi)$. Note also that $\widehat{\phi}(\xi) = 0$ for any ξ in this interval that is congruent mod 2π to $\pm\frac{2\pi}{7}$, $\pm\frac{4\pi}{7}$, or $\pm\frac{6\pi}{7}$.

From the consistency equation, we see that the complementary multiplicity function $\widetilde{m}$ only takes on the values 0 and 1, and in fact that $\widetilde{m}(\omega) = 1$ for $\omega \in [-\frac{2\pi}{7}, \frac{2\pi}{7}) \cup \pm[\frac{4\pi}{7}, \pi)$. In accordance with Theorem 2.5, we define a CCMF g as follows:

$$g(\omega) = \begin{cases} p(\omega+\pi) & \omega \in [0, \frac{\pi}{7}) \cup [\frac{3\pi}{7}, \frac{4\pi}{7}) \cup [\frac{6\pi}{7}, \pi) \\ -p(\omega+\pi) & \omega \in [-\pi, -\frac{6\pi}{7}) \cup [-\frac{4\pi}{7}, -\frac{3\pi}{7}) \cup [-\frac{\pi}{7}, 0) \\ \sqrt{2} & \omega \in \pm[\frac{2\pi}{7}, \frac{3\pi}{7}) \\ 0 & \text{otherwise} \end{cases}.$$

Note that g has jump discontinuities only at points that are congruent mod 2π to $\pm\frac{2\pi}{7}$ and $\pm\frac{6\pi}{7}$.

Because the discontinuities of the CCMF g are at $\pm\frac{2\pi}{7}$ and $\pm\frac{6\pi}{7}$ mod 2π, it follows that $\widehat{\psi}$, which is given by $\widehat{\psi}(\xi) = \frac{1}{\sqrt{2}}g(\frac{\xi}{2})\widehat{\phi}(\frac{\xi}{2})$, is C^∞ on the interval $(-2^{K+2}\pi, 2^{K+2}\pi)$.

Finally, to be sure that Corollary 3.7 applies, we must find a set B that satisfies the conditions of that corollary. This will require yet some further conditions on p. Namely, assume that there exists an $\epsilon > 0$ such that $p(\omega) = h'(\omega) = 0$ on $\pm(\frac{6\pi}{7} - \epsilon, \frac{6\pi}{7} + \epsilon) \cup (\pi - \epsilon, \pi + \epsilon)$, and that $h'(\omega) \neq 0$ for all points not previously mentioned. We set $B = \pm[\frac{6\pi}{7}, \frac{6\pi}{7} + \epsilon) \cup \pm(\frac{\pi}{7\times 2^K} - \frac{\epsilon}{2^{K+3}}, \frac{\pi}{7\times 2^K} + \frac{\epsilon}{2^{K+3}})$, and check directly that this B satisfies the requirements of Corollary 3.7.

EXAMPLE 4.3. The Journé wavelet is the inverse Fourier transform of the characteristic function of the set $[\frac{-32\pi}{7}, -4\pi) \cup [-\pi, \frac{-4\pi}{7}) \cup [\frac{4\pi}{7}, \pi] \cup [4\pi, \frac{32\pi}{7})$, and has multiplicity function

$$m(\omega) = \begin{cases} 2 & \omega \in [-\frac{2\pi}{7}, \frac{2\pi}{7}) \\ 1 & \omega \in \pm[\frac{2\pi}{7}, \frac{4\pi}{7}) \cup \pm[\frac{6\pi}{7}, \pi) \\ 0 & \text{otherwise} \end{cases},$$

so that $S_1 = [-\pi, \frac{-6\pi}{7}) \cup [\frac{-4\pi}{7}, \frac{4\pi}{7}) \cup [\frac{6\pi}{7}, \pi)$ and $S_2 = [\frac{-2\pi}{7}, \frac{2\pi}{7})$. Note that S_1 coincides with the set S of the preceding example.

We wish to construct an orthonormal wavelet ψ, whose associated multiplicity function is the Journé multiplicity function m, and whose Fourier transform is as smooth as possible. We will use the constructions in Example 4.2.

The canonical GCMF h described in Theorem 3.6 is given by the 2π periodizations of the following:

$$\begin{aligned} h_{1,1} &= \sqrt{2}\chi_{\frac{1}{2}S_1} \\ h_{2,1} &= \sqrt{2}\chi_{[-\pi, \frac{-6\pi}{7})\cup[\frac{6\pi}{7}, \pi)} \\ h_{1,2} &= h_{2,2} = 0. \end{aligned}$$

Note that $h_{1,1}$ coincides with the GCMF h in Example 4.2. As in that example, let the kernel element z_ω be π if $\omega \in [-\frac{\pi}{7}, \frac{\pi}{7}) \cup \pm[\frac{3\pi}{7}, \frac{\pi}{2})$ and 0 otherwise. Let p be the C^∞, periodic function of Example 4.2, and define a a new GCMF $h'_{i,j}$ for the Journé multiplicity function by

$$h'_{1,1}(\omega) = \begin{cases} \sqrt{2} & \omega \in \pm[\frac{\pi}{7}, \frac{2\pi}{7}) \\ p(\omega) & \omega \in [-\frac{\pi}{7}, \frac{\pi}{7}) \cup \pm[\frac{3\pi}{7}, \frac{4\pi}{7}) \cup \pm[\frac{6\pi}{7}, \pi) \\ 0 & \text{otherwise} \end{cases},$$

$$h'_{2,1}(\omega) = \begin{cases} p(\omega + \pi) & \omega \in [0, \frac{\pi}{7}) \cup [\frac{6\pi}{7}, \pi) \\ -p(\omega + \pi) & \omega \in [-\pi, -\frac{6\pi}{7}) \cup [-\frac{\pi}{7}, 0) \\ 0 & \text{otherwise} \end{cases},$$

and

$$h'_{1,2} = h'_{2,2} = 0.$$

Because the function $h_{1,1}$ agrees with the function h' of Example 4.2, we see that it is C^∞ except at points that are congruent to $\pm\frac{2\pi}{7}$ or $\pm\frac{4\pi}{7}$ mod 2π. It is clear that $h'_{2,1}$ is C^∞ except at points that are congruent to $\pm\frac{6\pi}{7}$ mod 2π.

The GCMF $\{h'_{i,j}\}$ is lower triangular, and the function $\widehat{\phi_1}$ coincides with the function $\widehat{\phi}$ of Example 4.2, and so is C^∞ on the interval $[-2^{K+1}\pi, 2^{K+1}\pi)$. The function $\widehat{\phi_2}$ is given by

$$\widehat{\phi_2}(\xi) = h'_{2,1}(\frac{\xi}{2})\widehat{\phi_1}(\frac{\xi}{2}),$$

showing that $\widehat{\phi_2}$ is C^∞ on the same interval, because $\widehat{\phi_1}$ is 0 where $h'_{2,1}$ has its discontinuities.

Now we will construct the Fourier transform of the wavelet ψ, by building a CCMF $\{g_{1,j}\}$. Again, following the construction in Theorem 2.5, we may take for $g_{1,1}$ the function defined by

$$g_{1,1}(\omega) = \begin{cases} p(\omega+\pi) & \omega \in [\frac{3\pi}{7}, \frac{4\pi}{7}) \\ -p(\omega+\pi) & \omega \in [-\frac{4\pi}{7}, -\frac{3\pi}{7}) \\ \sqrt{2} & \omega \in [\frac{2\pi}{7}, \frac{3\pi}{7}) \\ -\sqrt{2} & \omega \in [-\frac{3\pi}{7}, -\frac{2\pi}{7}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_{1,2}(\omega) = \begin{cases} \sqrt{2} & \omega \in [-\frac{2\pi}{7}, \frac{2\pi}{7}) \\ 0 & \text{otherwise} \end{cases}.$$

Note that the only points of discontinuity of $g_{1,1}$ and $g_{1,2}$ are points congruent to $\pm\frac{2\pi}{7}$ mod 2π. Therefore $\widehat{\psi}$, which is given by

$$\widehat{\psi}(\xi) = \frac{1}{\sqrt{2}}\left(g_{1,1}(\frac{\xi}{2})\widehat{\phi}_1(\frac{\xi}{2}) + g_{1,2}(\frac{\xi}{2})\widehat{\phi}_2(\frac{\xi}{2})\right),$$

is C^∞ on the interval $[-2^{K+2}\pi, 2^{K+2}\pi)$.

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