GENERALIZED MULTIRESOLUTION ANALYSES, AND A CONSTRUCTION PROCEDURE FOR ALL WAVELET SETS IN $\mathbb{R}^n$.

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Abstract. An abstract formulation of generalized multiresolution analyses is presented, and those GMRA's that come from multiwavelets are characterized. As an application of this abstract formulation, a constructive procedure is developed, which produces all wavelet sets in $\mathbb{R}^n$ relative to an integral expansive matrix.

Introduction

In 1996, Dai, Larson and Speegle ([DLS1]) proved the existence of wavelet sets in $\mathbb{R}^n$. That is, they proved that there exist sets $W \subseteq \mathbb{R}^n$ for which the indicator function χ_W is the Fourier transform of a (single) wavelet $\psi \in L^2(\mathbb{R}^n)$. This surprising result will undoubtedly have important applications, for in many cases it allows a reduction from multiwavelets to a single wavelet. Explicit examples of such wavelet sets in $\mathbb{R}^n$ have since been given by Dai, Larson, and Speegle [DLS2], Soardi and Weiland [SW], and others. In Section 2 of the present paper, we present a constructive procedure that produces all wavelet sets in $\mathbb{R}^n$ for which the expansive matrix A is integral. The construction is made using a pair of maps T and T' that are in a sense complementary with respect to the dilation matrix A.

In Section 3, we present several examples of wavelet sets constructed in this manner, and produce some new wavelet sets in $\mathbb{R}^1$, which generalize to give new wavelet sets in all dimensions. In addition we indicate how all previously known wavelet sets can be constructed using our technique.

Our construction comes from a theoretical investigation of wavelets and multiresolution analyses that concentrates on the unitary representation of the group of translations determined by its action on a fundamental invariant subspace. In Section 1, we develop a notion of generalized multiresolution analysis (GMRA), and identify which GMRA's determine multiwavelets. The criterion turns out to be a consistency equation that the multiplicity function associated to the representation satisfies. It is this consistency equation that we exploit in Section 2 for our constructive procedure.

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1. Generalized Multiresolution Analyses and Orthonormal Multiwavelets

We present here a notion of generalized multiresolution analyses (GMRA's), which enables us to study all multiwavelets, including those that do not arise from scaling functions. In fact, we obtain a correspondence between the set of all multiwavelets and a specified subset of the set of all GMRA's.

Let H be a separable Hilbert space. Fix a countable abelian group Γ of unitary operators on H, which we call *translations.*, and a unitary operator δ on H, which we call a *dilation.* We assume that the dilation is commensurate with the translations in the sense that the group $\delta^{-1}\Gamma\delta$ is a subgroup of finite index d in Γ.

In the main example of this paper, we will take H to be $L^2(\mathbb{R}^n)$ and Γ to be the group of translations by elements of the lattice $\mathbb{Z}^n$. Let A be an $n \times n$ integer dilation matrix such that all the eigenvalues of A have absolute value greater than 1. Finally, write δ_A for the unitary operator on $L^2(\mathbb{R}^n)$ given by

$$[\delta_A(f)](x) = |\det A|^{\frac{1}{2}} f(Ax).$$

We see that $\delta_A^{-1}\Gamma\delta_A$ is a subgroup of Γ of finite index $d = |\det A|$; thus Γ and δ_A satisfy the abstract assumptions above.

DEFINITION. By a *generalized multiresolution analysis* (GMRA) of H, relative to Γ and δ, we shall mean a collection $\{V_j\}_{-\infty}^{\infty}$ of closed subspaces of H that satisfy:

(1) $V_j \subseteq V_{j+1}$ for all j.
(2) $\delta(V_j) = V_{j+1}$ for all j.
(3) $\cup V_j$ is dense in H and $\cap V_j = \{0\}$.
(4) V_0 is invariant under the action of Γ.

If the closure of $\cup V_j$ is a subspace of H, we say that the collection $\{V_j\}$ is a *subspace GMRA*.

The classical definition of an MRA includes the assumption that there exists a scaling vector ϕ whose translates form an orthonormal basis for V_0. While a GMRA will not in general have a scaling vector, we can obtain similar information about its structure by studying the unitary representation ρ determined by the action of Γ on V_0. We refer to V_0 as the *core subspace* and the representation ρ as the *core representation* of Γ relative to the generalized multiresolution analysis $\{V_j\}$. In the MRA case, the core representation is equivalent to the regular representation Λ of Γ, which acts in $L^2(\Gamma)$ by translation. (The equivalence is effected by mapping the scaling function ϕ to the characteristic function of the identity in $L^2(\Gamma)$.) As we will see below, a variety of other core representations are possible, and in fact those GMRA's that arise from wavelets can be characterized in terms of their core representations.

For each $j \in \mathbb{Z}$, we write W_j for the orthogonal complement of V_j in V_{j+1}. We have then that

$$H = \bigoplus_{j=-\infty}^{\infty} W_j = V_0 \oplus \bigoplus_{j=0}^{\infty} W_j.$$

For $j \geq 0$, each V_j and therefore each W_j is invariant under Γ. Just as in the MRA case, the decomposition $V_1 = V_0 \oplus W_0$ will be particularly useful in relating GMRA's to wavelets. Accordingly, we let ρ' and σ denote the unitary representations of Γ determined by its action on V_1 and W_0 respectively.

We next employ some tools from abstract harmonic analysis. By the spectral theorem for commutative groups, the representation ρ can be decomposed as a direct integral over $\widehat{\Gamma}$, the group of characters of Γ. (See e.g. [M], [Ha], [He].) More precisely, there exists a unique projection-valued measure p on $\widehat{\Gamma}$ for which

$$\rho_\gamma = \int_{\widehat{\Gamma}} \gamma(\chi)\, dp(\chi).$$

The representation ρ' is similarly determined by a projection-valued measure p' on $\widehat{\Gamma}$.

We will analyze ρ by letting Γ act on both sides of the decomposition $V_1 = V_0 \oplus W_0$. To relate the actions of Γ on V_0 and V_1, we use conjugation by δ. Accordingly, let $\alpha : \Gamma \mapsto \Gamma$ by $\alpha(\gamma) = \delta^{-1}\gamma\delta$. Write α^* for the dual map on $\widehat{\Gamma}$, given by $[\alpha^*(\chi)](\gamma) = \chi(\alpha(\gamma))$. Finally, let $\alpha^*_*(p)$ denote the projection-valued measure on $\widehat{\Gamma}$ given by $\alpha^*_*(p)(E) = p(\alpha^{*-1}(E))$.

PROPOSITION 1.1. *The representation ρ' of Γ satisfies*

$$\rho_\gamma \oplus \sigma_\gamma \equiv \rho'_\gamma \equiv \int_{\widehat{\Gamma}} \gamma(\chi)\, d[\alpha^*_*(p)](\chi).$$

PROOF. By writing $V_1 = V_0 \oplus W_0$, we see immediately that ρ' restricted to Γ is equivalent to the direct sum $\rho \oplus \sigma$. Since $\delta^{-1}\rho'_\gamma\delta = \rho_{\alpha(\gamma)}$, the two representations ρ' and $\rho \circ \alpha$ of Γ are unitarily equivalent. Thus the projection valued measure p' is unitarily equivalent to $\alpha^*_*(p)$. The equivalence of ρ' to the direct integral above then follows. □

By the spectral multiplicity theory developed by Stone [S] and Mackey [M] (see also [He] and [Ha]), the projection-valued measure p is completely determined by a measure class $[\mu]$ on $\widehat{\Gamma}$, and a multiplicity function m mapping $\widehat{\Gamma}$ into the set $\{0, 1, 2, \ldots, \infty\}$. This multiplicity function roughly counts the number of times each character occurs in the representation ρ.

DEFINITION. We call the multiplicity function m of the preceding paragraph the *core multiplicity function* corresponding to the GMRA $\{V_j\}$ and the measure class of $[\mu]$ the *core measure class*.

Note that a GMRA is an MRA if and only if its core measure class is that of Haar measure and the core multiplicity function m is 1 almost everywhere. This follows from our definitions since Haar measure and $m \equiv 1$ is equivalent to ρ being the regular representation of Γ. GMRA's with other multiplicity functions and measure classes provide a tool for studying the wide range of wavelets that are not associated with MRA's.

DEFINITION. By a *multiwavelet* for H relative to Γ and δ, we mean a finite collection $\{\psi_1, \ldots, \psi_N\}$ of vectors in H such that the collection $\{\delta^j(\gamma(\psi_i))\}$, for $j \in \mathbb{Z}$, $\gamma \in \Gamma$, and $1 \le i \le N$, forms an orthonormal basis for H. The collection $\{\psi_1, \ldots, \psi_N\}$ is called a *subspace multiwavelet* if these vectors form an orthonormal basis for a subspace of H.

PROPOSITION 1.2. *Let $\{\psi_1, \ldots, \psi_N\}$ be a multiwavelet for H, relative to Γ and δ, and set V_j equal to the closed linear span of the vectors $\delta^k(\gamma(\psi_i))$, for $k < j$, $\gamma \in \Gamma$, and $1 \leq i \leq N$. Then the collection $\{V_j\}$ is a generalized multiresolution analysis of H. Moreover, if $H = L^2(\mathbb{R}^n)$, $\Gamma = \mathbb{Z}^n$ and $\delta = \delta_A$, then the core measure class is absolutely continuous with respect to Haar measure.*

PROOF. The fact that the collection of subspaces $\{V_j\}$ meet the requirements for a GMRA (for any H, Γ, and δ) is well known and straightforward (see for example [BCMO].) Thus it remains to be shown that the core measure class is absolutely continuous with respect to Haar measure under the special assumptions on H, Γ, and δ.

By taking Fourier transforms, the representation of the group $\Gamma = \mathbb{Z}^n$ on all of $H = L^2(\mathbb{R}^n)$ is equivalent to multiplication by exponentials $e^{i\langle x,\gamma\rangle}$ on $L^2(\mathbb{R}^n)$. By parametrizing $\widehat{\Gamma} = [-\pi, \pi)^n$ with addition mod 2π, the regular representation of Γ can be seen to be equivalent to multiplication by exponentials on $L^2([-\pi, \pi)^n)$. Thus, by writing $\mathbb{R}^n$ as the disjoint union of translates of the cube $[-\pi, \pi)^n$ by elements of Γ, we have that the representation of Γ on all of H is equivalent to an infinite multiple of the regular representation of Γ. Therefore, the projection-valued measure associated to the representation of Γ on all of H is equivalent to Haar measure on the dual $\widehat{\Gamma}$, so that the projection-valued measure associated to any subrepresentation of this representation will have a corresponding measure that is absolutely continuous with respect to Haar measure. $\square$

We see then that those GMRA's that arise from multiwavelets in $\mathbb{R}^n$ have core representations that are completely determined by their core multiplicity function m. The main theoretical result of this paper is the following theorem, which uses the multiplicity function to describe precisely how the two notions of multiwavelets and generalized multiresolution analyses are related. Let μ be a representative of the core measure class, and λ be Haar measure on $\widehat{\Gamma}$.

THEOREM 1.3. *If $\{\psi_1, \ldots, \psi_N\}$ is a multiwavelet for a Hilbert space H then the collection of subspaces $\{V_j\}$ determined by the ψ_i's as in Proposition 1.2 is a generalized multiresolution analysis, whose core multiplicity function m satisfies the following consistency equation almost everywhere with respect to the measure $\mu + \lambda$:*

$$(*) \qquad m(\chi) + N = \sum_{\phi \in \alpha^{*-1}(\chi)} m(\phi).$$

Conversely, if $\{V_j\}$ is a generalized multiresolution analysis of a Hilbert space H whose core multiplicity function m is finite almost everywhere and satisfies the consistency equation $(*)$, almost everywhere with respect to the measure $\mu + \lambda$, then there exist vectors $\psi_1, \ldots, \psi_N$ in the subspace W_0 that form a multiwavelet for H. Moreover, the GMRA $\{V_j\}$ coincides with the GMRA determined from these ψ_i's as in Proposition 1.2. The analogous statements hold for subspace GMRA's and subspace multiwavelets.

PROOF. Suppose first that $\psi_1, \ldots, \psi_N$ is a multiwavelet for H. Then, we have seen in Proposition 1.2 that the subspaces $\{V_j\}$ determined by the ψ_i's form a GMRA. We have also seen in the proof of Proposition 1.1 that $\rho' \equiv \rho \oplus N \times \Lambda$, and also that $\rho'_\gamma \equiv \int_{\widehat{\Gamma}} \gamma(\chi)\, d[\alpha^*_*(p)](\chi)$. Thus, the measure class of ρ' must be that of $\mu + \lambda$, and the multiplicity functions for the two characterizations of ρ' give exactly the two sides of the consistency equation $(*)$.

Conversely, suppose that $\{V_j\}$ is a GMRA whose multiplicity function m is finite almost everywhere and satisfies the consistency equation $(*)$. It follows from the consistency equation that ρ' is equivalent to $\rho \oplus N \times \Lambda$. By writing $V_1 = V_0 \oplus W_0$, we also have that ρ' is equivalent to $\rho \oplus \sigma$. Because the multiplicity function m is finite almost everywhere, the representation ρ' generates a finite von Neumann algebra, and this implies that $\sigma \equiv N \times \Lambda$; that is we may cancel the direct summand ρ. (See Section 1 of [BCMO] for a similar argument.) Therefore, there must exist vectors $\psi_1, \dots, \psi_N \in W_0$ such that the vectors $\{\gamma(\psi_i)\}$, for $\gamma \in \Gamma$ and $1 \leq i \leq N$, form an orthonormal basis for W_0. Since the operator δ is unitary, and since the spaces $\{\delta^j(W_0)\}$ are orthogonal and span all of H, it follows that the ψ_i's form a multiwavelet. It is immediate that the GMRA determined by the vectors $\psi_1, \dots, \psi_N$ coincides with the given GMRA. $\square$

2. Construction of Wavelet Sets

We now explore some applications of the theoretical results of Section 1 to our main example of $H = L^2(\mathbb{R}^n)$, $\Gamma = \mathbb{Z}^n$, and $\delta = \delta_A$. We will be particularly interested in studying single wavelets determined by wavelet sets. Following Dai and Larson [DL], we define a *wavelet set* to be a set $W \subset \mathbb{R}^n$ such that $\chi_W = \hat{\psi}$, where ψ is a (single) wavelet for $\mathbb{R}^n$. By a subspace wavelet set, we will mean a set $W \subset \mathbb{R}^n$ such that $\chi_W = \hat{\psi}$ for a subspace wavelet ψ. The first Theorem of this section translates the abstract requirement that the core multiplicity function associated with a wavelet satisfies a consistency equation into this Euclidean space setting.

We write $x \equiv y$ if $x - y = 2\pi j$ for some $j \in \mathbb{Z}^n$. Write Q for the cube $[-\pi, \pi)^n$, and for $x \in \mathbb{R}^n$, let $\bar{x}$ denote the unique element of Q such that $\bar{x} \equiv x$. We write A^* for the transpose of the matrix A.

THEOREM 2.1. *The set $W \subset \mathbb{R}^n$ is a subspace wavelet set if and only if the indicator function χ_E of the set $E = \cup_{j<0}(A^*)^j(W)$ satisfies the following consistency equation:*

$$(**) \qquad 1 + \sum_{k \in \mathbb{Z}^n} \chi_E(x + 2\pi k) = \sum_{j \in \mathbb{Z}^n} \chi_E((A^*)^{-1}(x + 2\pi j))$$

for almost all $x \in \mathbb{R}^n$. In particular, W is a wavelet set for all of $\mathbb{R}^n$ if and only if in addition, $\cup_{j \in \mathbb{Z}}(A^*)^j(E)$ contains, up to a set of measure 0, a neighborhood of the origin.

PROOF. Let $\{V_j\}$ denote the (subspace) GMRA determined by the (subspace) wavelet ψ whose Fourier transform is χ_W. Then, since $V_0 = \bigoplus_{j<0} \delta^j(W_0)$, we see that $\widehat{V_0}$ coincides with $L^2(E)$. Moreover, it follows from Section 1 that the Fourier transform of the core representation ρ of the group $\mathbb{Z}^n$ is given by multiplication on $L^2(E)$ as follows:

$$[\rho_k(f)](x) = e^{i\langle k, x\rangle} f(x),$$

so that the core multiplicity function m on $[-\pi, \pi)^n$ is given by

$$m(x) = \sum_{k \in \mathbb{Z}^n} \chi_E(x + 2\pi k).$$

The sum on the right-hand side of the consistency equation $(*)$ is over all $y \in [-\pi, \pi)^n$ such that $< y, Aj > \equiv < x, j >$ mod 2π for all $j \in \mathbb{Z}^n$. The $\mathbb{R}^n$ case of the consistency equation $(**)$ then follows.

If ψ is a wavelet for all of $L^2(\mathbb{R}^n)$, then the union of the dilates of W is, by [DLS1], all of $\mathbb{R}^n$. Since the union of the dilates of E (by A^*) contains the union of the dilates of W, it follows that $\cup_{j\in\mathbb{Z}}(A^*)^j(E)$ contains a neighborhood of 0.

Conversely, given a set W such that $E = \cup_{j<0}(A^*)^j(W)$ satisfies the consistency equation $(**)$, we define a (subspace) GMRA as follows: Define V_j by setting $\widehat{V_j}$ to be $\delta^{-j}L^2(E)$. We have that $\{V_j\}$ is a GMRA, and $L^2(W) = \widehat{W_0}$. As above, we see that the core multiplicity function m for this GMRA is given by

$$m(x) = \sum_{k\in\mathbb{Z}^n} \chi_E(x + 2\pi k).$$

The requirement that χ_E satisfy the consistency equation $(**)$ says exactly that m satisfies the consistency equation $(*)$ of Section 1, and so by Theorem 1.3 this GMRA determines a single (subspace) wavelet with Fourier transform χ_W.

Since E is itself a union of dilates of W, it follows immediately that if $\cup_{j\in\mathbb{Z}}(A^*)^j(E)$ contains a neighborhood of 0, then $\cup_{j\in\mathbb{Z}}(A^*)^j(W)$ contains a neighborhood of 0. It follows then from [DLS1] that W is a wavelet set for all of $\mathbb{R}^n$. $\square$

We will use Theorem 2.1 to give an explicit technique for constructing all wavelet sets in $\mathbb{R}^n$ by building sets E whose indicator functions satisfy the consistency equation $(**)$. The construction is based on the following definition.

DEFINITION. Let E be a subset of $\mathbb{R}^n$ that is invariant under $(A^*)^{-1}$. By an *A-complementary pair for E* we mean a pair (T, T') of measurable one-to-one maps $T : Q \to \mathbb{R}^n$ and $T' : E \to E$ satisfying (up to a set of measure 0):

(1) $T(Q) \subseteq E$, and $T'(E) \subseteq E \setminus T(Q)$.
(2) $A^*(T(x)) \equiv x$ and $A^*(T'(x)) \equiv x$.
(3) $E = \cup_{j\geq 0} T'^j(T(Q))$.

THEOREM 2.2. *If (T, T') is an A-complementary pair for a set $E \subseteq \mathbb{R}^n$, then $W = A^*(E) \setminus E$ is a (subspace) wavelet set. Conversely, if $W \subseteq \mathbb{R}^n$ is a (subspace) wavelet set, then there exists an A-complementary pair (T, T') for the set $E = \cup_{j<0}(A^*)^j(W)$. In fact, T' can always be taken equal to $(A^*)^{-1}$.*

PROOF. Suppose W is a subspace wavelet set. Then $E = \cup_{k<0}(A^*)^k(W)$ is in fact a disjoint union [DL], and is invariant under $(A^*)^{-1}$. By Theorem 2.1, χ_E must satisfy the consistency equation $(**)$. We will use these facts to define an A-complementary pair of maps T and T' for E.

It follows from $(**)$, and the fact that the multiplicity function m is finite almost everywhere, that for each $x \in \mathbb{R}^n$, there exists a unique point w of the form $(A^*)^{-1}(x + 2\pi j)$ such that w belongs to E but $A^*(w) = x + 2\pi j$ itself does not belong to E. For each $x \in Q$, let $T(x)$ be this unique point $w = (A^*)^{-1}(x + 2\pi j)$. Define T' on E by $T'(x) = (A^*)^{-1}(x)$. The maps T and T' are one-to-one and satisfy condition (2) of the definition of A-complementary pair.

To establish condition (3), it will suffice to show that $T(Q) = (A^*)^{-1}(W)$. First, we take $x \in Q$ and show that $T(x) \in (A^*)^{-1}(W)$. Indeed, $T(x) \in E$ by definition, and it is not of the form $(A^*)^{-1}(w)$ for any $w \in E$. Thus $T(x)$ is in $\cup_{j<0}(A^*)^j(W) \setminus \cup_{j<0}(A^*)^{j-1}(W)$ and hence (since the union is disjoint), in $(A^*)^{-1}(W)$. Next, we take $y \in (A^*)^{-1}(W)$ and show that $y \in T(Q)$. Let $x = A^*(y)$, and note, again by the disjointness of the dilates of W, that $x \notin E$. Recall that $T(\bar{x})$ is the unique point in E of the form $(A^*)^{-1}(x + 2\pi j)$ where $x + 2\pi j \notin E$. Hence $T(\bar{x})$ must be $(A^*)^{-1}(x) = y$, so that y does belong to $T(Q)$. Finally, the equality just established of $T(Q) = (A^*)^{-1}(W)$ also implies condition (1). We have thus verified that the pair (T, T') is an A-complementary pair for E.

Conversely, let (T, T') be an A-complementary pair for the set $E = \cup_{j \geq 0} T'^j(T(Q))$. We show first that χ_E satisfies the consistency equation $(**)$. Note that for each $x \in E$, there is exactly one element of the form $(A^*)^{-1}(x+2\pi j)$ that is in $T(Q)$, namely $T(\bar{x})$. For, if $z \in T(Q)$ is of the form $(A^*)^{-1}(x + 2\pi j)$, then $z = T(w)$ for some $w \in Q$, implying that $A^*(z) \equiv w$. But $A^*(z) \equiv x$, and this implies that $w = \bar{x}$, so that $z = T(\bar{x})$ as claimed.

For each $x \in \mathbb{R}^n$, this unique element of the form $(A^*)^{-1}(x + 2\pi j)$ in $T(Q)$ contributes 1 to the sum on the right side of the consistency equation, which can be used to cancel the (extra) 1 on the left hand side, so that the equation can be modified to become

$$\sum_{k \in \mathbb{Z}^n} \chi_E(x + 2\pi k) = \sum_{j \in \mathbb{Z}^n} \chi_{E \setminus T(Q)}((A^*)^{-1}(x + 2\pi j)).$$

For any $x \in \mathbb{R}^n$, if $y = (A^*)^{-1}(x + 2\pi j) \in E \setminus T(Q)$, then $y = T'^m(z)$ for some $z \in T(Q)$ and some $m > 0$. So, $x \equiv A^*(y) \equiv T'^{m-1}(z)$, which shows that $x + 2\pi k \in E$ for some $k \in \mathbb{Z}^n$. Thus, if the left hand side of the modified consistency equation is 0, the right hand side must be 0 as well. If not, let $x_1, x_2, \ldots$ be a complete list of the elements of the form $x+2\pi k$ that are in E. Then $T'(x_1), T'(x_2), \cdots$ are distinct elements of $E \setminus T(Q)$. Each of $T'(x_1), T'(x_2), \ldots$ is of the form $(A^*)^{-1}(x + 2\pi j)$. And, if z is any element of this form in $E \setminus T(Q)$, then $z = T'(w)$ for some $w \equiv x$, so that z is one of the $T'(x_i)$'s we have listed. Thus the nonzero terms on the left hand side of the modified consistency equation are in one-to-one correspondence with the nonzero terms on the right, and so the consistency equation is satisfied.

Because E is invariant under $(A^*)^{-1}$, we have that $E \subseteq A^*(E)$. Let $W = A^*(E) \setminus E$. The proof will be complete, by Theorem 2.1, if we can show that, up to a set of measure 0, $E = \cup_{k<0}(A^*)^k(W)$. It is evident that this union is contained in E. Moreover,

$$\begin{aligned} E \setminus \cup_{k<0}(A^*)^k(W) &= E \setminus \cup_{k<0}(A^*)^k(A^*(E) \setminus E) \\ &= \cap_{k<0}(A^*)^k(E). \end{aligned}$$

We know by condition (2) of the definition that for any measurable set $F \subseteq E$, $\mu(T'(F)) = \frac{1}{|\det A|}\mu(F)$. Hence $E = \cup_{j \geq 0} T'^j(T(Q))$ must have finite measure. It then follows that $\cap_{k<0}(A^*)^k(E)$ has measure 0. □

We now prove that every map $T : Q \to \mathbb{R}^n$ of the form $T(x) = (A^*)^{-1}(x+2\pi j_x)$ where for each x, $j_x \in \mathbb{Z}^n$, is part of an A-complementary pair. In the process, we describe a method for constructing A-complementary pairs and hence, via the two preceding theorems, wavelet sets. We will explicitly carry out this kind of construction in the next section.

THEOREM 2.3. *If T is any measurable map of Q into $\mathbb{R}^n$ satisfying $A^*(T(x)) \equiv x$ for almost all x, then there exists a set $E \subseteq \mathbb{R}^n$ and a map $T' : E \to E$ such that (T, T') is an A-complementary pair for E.*

PROOF. We will define T' first on $E_0 \equiv T(Q)$, and then recursively on $E_1, E_2, \cdots$, where $E_m = \cup_{j=0}^{m} T'^j(T(Q)) \setminus \cup_{j=0}^{m-1} T'^j(T(Q)),\ m > 0$. In order to ensure that the resulting pair (T, T') will be an A-complementary pair for $E = \cup_{m=0}^{\infty} E_m = \cup_{j=0}^{\infty} T'^j(T(Q))$, it will suffice to define T' such that for each $x \in E_m$:

(i) If $(A^*)^{-1}(x) \notin \cup_{j=0}^{m} E_j$ then $T'(x) = (A^*)^{-1}(x)$.
(ii) $A^*(T'(x)) \equiv x$.
(iii) $T'(x) \neq T(\bar{x})$.
(iv) $T'(x) \neq T'(x + 2\pi l)$ if $l \neq 0$ and $x + 2\pi l \in \cup_{j=0}^{m} E_j$.

For, then E will be invariant under $(A^*)^{-1}$ by (i), and T' will be a one-to-one map by (iv). Condition (1), (2), and (3) of the definition of an $A-$complementary pair will follow from (iii), (ii), and the definition of E, respectively.

We will use an induction argument to prove that it is always possible to define T' in this way. Thus, fix $m \geq 0$, and suppose that T' has already been defined to satisfy (i)-(iv) on $\cup_{j=0}^{m-1} E_j$. (In the $m = 0$ case, this already defined set is empty.) We must now define $T'(x)$ for $x \in E_m$. We begin by defining $T'(x)$ to be $(A^*)^{-1}(x)$ whenever condition (i) applies. Condition (ii) merely requires that $T'(x)$ be one of the infinite number of elements in the set $C(x) \equiv \{(A^*)^{-1}(x + 2\pi j) : j \in \mathbb{Z}^n\}$. Observe that, although condition (iii) eliminates one element of $C(x)$ as a possible choice for $T'(x)$, it does not conflict with condition (i), since if $(A^*)^{-1}(x) \notin E \supseteq E_0 = T(Q)$, it is impossible for $(A^*)^{-1}(x)$ to equal $T(\bar{x})$. Condition (iv) also does not conflict with condition (i), since $(A^*)^{-1}$ is one-to-one. Thus, to complete the definition of T' on E_m, it will suffice to show that, when condition (i) does not apply, we can choose $T'(x)$ to be an element of the set $C(x) \setminus \{T(\bar{x})\}$ in accordance with condition (iv). We make our definition simultaneously for all $y \in E_m$such that $y \equiv x$. Note that, because the measure of each E_j is finite, we have that for almost all x, $C(x) \cap E_j$ is finite, and hence only a finite number of elements of $C(x)$ can have already been used as $T'(z)$ for $z = (x + 2\pi k)$ and $z \in \cup_{j=0}^{m-1} E_j$. We thus finish the proof by defining $T'(x + 2\pi k)$ for $x + 2\pi k \in E_m$ to be arbitrary distinct elements chosen from the infinite set $C(x) \setminus (\{T(\bar{x})\} \cup (C(x) \cap (\cup_{j=0}^{m-1} E_j))$. $\square$

3. Examples

In this section we use the method indicated in Theorems 2.2 and 2.3 to construct a variety of wavelet sets. That is, we use the ideas from Theorem 2.3 to define A-complementary pairs (T, T'), and then follow the construction outlined in the proof of Theorem 2.2 to build the associated wavelet sets W. Our procedure is in some ways similar to the theoretical construction of wavelet sets in [DLS1], as well as to that in [SW].

Theorem 2.2 shows that an arbitrary $A-$complementary pair gives a wavelet set, even though all wavelet sets can be defined using only A-complementary pairs that have $T' = (A^*)^{-1}$. This redundancy sometimes allows for simpler and more natural constructions. In particular, the easiest way to build an $A-$complementary pair is to start by defining $T(x) = (A^*)^{-1}(x)$ for $x \in Q$. We use this technique in our first example, which builds many of the known wavelet sets as well as some similar new ones.

EXAMPLE 3.1. We first use $T(x) = (A^*)^{-1}(x)$ to construct some wavelet sets in $\mathbb{R}^1$ for dilation by 2, and then indicate the analogous constructions in higher dimensions, and for other dilations.

(i) We start with the map T defined by $T(x) = \frac{x}{2}$ for all $x \in Q = [-\pi, \pi)$. We then define T', in accordance with Theorem 2.3, by $T'(x) = \frac{x}{2} + \pi$ for $x < 0$ and $T'(x) = \frac{x}{2} - \pi$ for $x \geq 0$. This results in $E = \cup_{j \geq 0} T'^{j} T([-\pi, \pi)) = [-\pi, \pi)$, so that $W = 2E \setminus E$ is the Shannon wavelet set.

(ii) Other simple wavelet sets for dilation by 2 in $\mathbb{R}$ can be built similarly. For example, with the same definition of T, we fix $n \geq 0$ and define $T'_n(x) = \frac{x}{2}$ for $x \notin Q$, and otherwise by $T'_n(x) = \frac{x}{2} - 2^n\pi$ for $-\pi \leq x < 0$ and $T'_n(x) = \frac{x}{2} + 2^n\pi$ for $0 \leq x < \pi$. The resulting wavelet set is:

$$W_n = [-2^{n+1}\pi - \alpha_n, -2^{n+1}\pi) \cup [-\pi, -\alpha_n) \cup [\alpha_n, \pi) \cup [2^{n+1}\pi, 2^{n+1}\pi + \alpha_n),$$

where $\alpha_n = \frac{2^{n+1}\pi}{2^{n+2}-1}$. The case $n = 0$ appears in Example 4.5 (iv) in [DL]. The case $n = 1$ is an example due to Journe (see e.g. Example 4.1 (i) in [DL]). Only the $n = 0$ example comes from an MRA, since none of the other multiplicity functions are identically 1. In particular, $m_n(x) = n + 1$ on $[-\frac{\alpha_n}{2^n}, \frac{\alpha_n}{2^n})$.

(iii) We now consider some analogous constructions in $\mathbb{R}^2$ (which easily generalize to higher dimensions). Define T on $Q = [-\pi, \pi) \times [-\pi, \pi)$ by setting $T(x) = \frac{x}{2}$ for all $x \in Q$. The simplest definitions for T' result from first defining T' in the first quadrant, and then using some symmetry requirement to extend it. If we require T' to be symmetric with respect to both axes, the first quadrant map $T'(x) = \frac{x}{2} + (\pi, \pi)$ if $x \in Q$ and $T'(x) = \frac{x}{2}$ if $x \notin Q$ yields the wavelet set of Dai, Larson, and Speegle that they call the "four corners"([DLS2]). Replacing (π, π) by $(-\pi, -\pi)$ yields Soardi and Weiland's first example ([SW]); replacing (π, π) instead by $(\pi, 0)$ yields the wavelet set Dai, Larson, and Speegle call the "wedding cake"set ([DLS2]). If we change the symmetry requirement to 4-fold rotational symmetry, and define $T'(x) = \frac{x}{2} + (-\pi, 0)$ if $x \in Q$ and $T'(x) = \frac{x}{2}$ if $x \notin Q$, we get the "windmill" set pictured below.

(iv) Any of the examples above can be generalized to work for other dilations. However, in many examples, generalizations to different dilations lead to wavelet sets with very different shapes than in the $d = 2$ case. For example, to apply (ii) to a dilation by d on $\mathbb{R}$, we can again fix $n \geq 0$ and take $T(x) = \frac{x}{d}$ on Q. We define $T'_n(x) = \frac{x}{d}$ for $x \notin Q$, and otherwise, $T'_n(x) = \frac{x}{d} - d^{n-1}2\pi$ for $-\pi \leq x < 0$ and $T'_n(x) = \frac{x}{d} + d^{n-1}2\pi$ for $0 \leq x < \pi$. The wavelet sets for $d \neq 2$ have a Cantor-like shape. For example, for $d = 3$, if we define the map $S_n : \mathbb{R} \mapsto \mathbb{R}$ by $S_n(x) = \frac{x}{3^{n+1}} + \frac{2\pi}{3}$ then we can describe the resulting wavelet sets by

$$W_n = \left(\cup_{j=0}^{\infty} S_n^j([\frac{\pi}{3}, \frac{2\pi}{3}))\right) \cup \left(\cup_{j=0}^{\infty} (S_n^j([0, \frac{\pi}{3})) + 3^n \cdot 2\pi)\right)$$

EXAMPLE 3.2. At the opposite extreme from Example 3.1, perhaps, is the following construction. This time we define $T(x)$ so that it is never equal to $(A^*)^{-1}(x)$. Again, we start with $\mathbb{R}^1$ and dilation by 2.

(i) To begin with, set $T(x) = \frac{x}{2} + \pi$ for $x \geq 0$ and $\frac{x}{2} - \pi$ for $x < 0$. Then $T'(x)$ may always be defined by $\frac{x}{2}$, and the resulting set E is the symmetric set whose intersection with the positive reals consists of the union of the intervals $[\frac{\pi}{2^k}, \frac{3\pi}{2^{k+1}})$, for $k \geq 0$. The resulting set $W = 2E \setminus E$ is then $W = [-3\pi, -2\pi) \cup [2\pi, 3\pi)$, which is only a subspace wavelet set, since it is **not** dilation congruent to the entire real line. Note that, as Theorem 2.1 asserts, the union of the dilates of E does not contain any neighborhood of 0. It is interesting to note also that the multiplicity function $m(x) = \sum_{k\in\mathbb{Z}} \chi_E(x+2\pi k)$ corresponding to the GMRA determined by this subspace wavelet set w takes on all three values 0,1, and 2. For instance, $m(x) = 2$ for $\frac{1}{2} < |x| < \frac{3}{4}$. Hence, the subspace GMRA determined by this subspace wavelet is not a subspace MRA.

(ii) If we alter our definition of T so that $T(x) = \frac{x}{2} - \pi$ if $x \geq 0$ and $\frac{x}{2} + \pi$ if $x < 0$, then again $T'(x)$ may always be defined as $\frac{x}{2}$, and the set E is the entire interval $[-\pi, \pi)$, whence the wavelet set is again the Shannon set.

(iii) For a more complicated example, consider the following parameterized family, which interpolates between the two previous ones. For $0 \leq \alpha \leq 1$, symmetrically define $T(x)$ as follows: For $0 \leq x < \pi$ set $T(x) = \frac{x}{2} + \pi$ if $x \geq \alpha\pi$ and $\frac{x}{2} - \pi$ if $x < \alpha\pi$. Then the set $T(Q)$ is the symmetric set whose positive part is $[(1 - \frac{\alpha}{2})\pi, \pi) \cup [(1 + \frac{\alpha}{2})\pi, \frac{3}{2}\pi)$. In accordance with the constructive procedure of Theorem 2.3, we define $T'(x)$ to be $\frac{x}{2}$ if $x < \pi$. However, we may not be able to set $T'(x) = \frac{x}{2}$ for every $x \geq \pi$. In fact, for $\alpha \geq \frac{1}{2}$, we see that $\frac{x}{2}$ is already in the set $T(Q)$ when $(2 - \alpha)\pi \leq x < \frac{3}{2}\pi$. (For $\alpha \leq \frac{1}{2}$, this troublesome set of x's is empty.) Accordingly, we set $T'(x) = \frac{x}{2} - \pi$ if $(2 - \alpha)\pi < x \leq \frac{3}{2}\pi$, and $T'(x) = \frac{x}{2}$ otherwise. When $\alpha \geq \frac{2}{3}$, the resulting set E_α contains the intervals $[\frac{1}{4}\pi, (\frac{1}{2} - \frac{\alpha}{4})\pi)$ and $[(\frac{1}{2} - \frac{\alpha}{4})\pi, \frac{1}{2}\pi)$, and hence the entire interval $[\frac{1}{4}\pi, \frac{1}{2}\pi)$, and therefore the union of the dilates of E_α contains a neighborhood of 0. The positive part of the resulting symmetric wavelet set W_α, for $\frac{2}{3} < \alpha$, is then

$$W_\alpha = [\frac{1}{2}\pi, (1 - \frac{\alpha}{2})\pi) \cup [(2 - \alpha)\pi, (1 + \frac{\alpha}{2})\pi) \cup [\frac{3}{2}\pi, 2\pi) \cup [(2 + \alpha)\pi, 3\pi).$$

The multiplicity functions for the GMRA's determined by the W_α's, for $\frac{2}{3} < \alpha < 1$, are identically 1, so that these GMRA's are MRA's. For $\alpha < \frac{2}{3}$, the set E_α is more complicated and its dilates do not contain a neighborhood of 0. However the sets W_α still are finite unions of intervals. If $\frac{1}{2} \leq \alpha < \frac{2}{3}$, we have

$$W_\alpha = [\frac{1}{2}\pi, \alpha\pi) \cup [\frac{3}{2}\pi, 2\pi) \cup [(2 + \alpha)\pi, 3\pi),$$

and for $0 \leq \alpha < \frac{1}{2}$, we have

$$W_\alpha = [(2 - \alpha)\pi, 2\pi) \cup [(2 + \alpha)\pi, 3\pi).$$

The multiplicity functions for all these GMRA's take on the value 2 some places. These W_α's, for $\alpha < \frac{2}{3}$, determine subspace wavelets, and, in fact, the limiting case W_0 is the wavelet described in (i) above. Also, W_1 is the wavelet set described in (ii). Therefore, there seems to be no simple continuity, as a function of α, of the property of being a subspace wavelet as opposed to a full space wavelet, nor of the multiplicity function m_α.

Analogous constructions in $\mathbb{R}^2$, e.g., defining $T(x) = \frac{x}{2} + (\pi, \pi)$ for x in the first quadrant, and so on, are possible. As in the 1-dimensional case, some constructions give subspace wavelet sets and others give wavelet sets.

EXAMPLE 3.3. More interesting and complicated wavelet sets can be constructed by intermingling further the two ways of defining T in the preceding examples, i.e., by defining $T(x)$ to be $\frac{x}{2}$ some of the time and not $\frac{x}{2}$ the rest of the time. For instance, the following constructions produce wavelet sets W that have 0 as a limit point.

(i) let $\{a_n\}_1^\infty$ be the sequence defined by $a_n = (\frac{1}{2} + \frac{1}{2^n})\pi$. That is, the sequence

$$\{a_n\} = \{\pi, \frac{3}{4}\pi, \frac{5}{8}\pi, \frac{9}{16}\pi, \dots\}.$$

Define $T(x) = \frac{x}{2}$ for all

$$x \in \cup_{n=1}^\infty \cup_{j=n}^\infty \frac{1}{2^{j+1}}[a_{n+1}, a_n).$$

For other $x \geq 0$, set $T(x) = \frac{x}{2} - \pi$, and for all $x < 0$, set $T(x) = \frac{x}{2}$. One sees that the intersection of the set $T(Q)$ with the negative half line contains the interval $[-\frac{5}{8}\pi, 0)$, and that the intersection of $T(Q)$ with the positive half line coincides precisely with $\cup_{n=1}^\infty \cup_{j=n}^\infty \frac{1}{2^{j+1}}[a_{n+1}, a_n)$. Therefore, the union of the dilates of $T(Q)$ contains each interval $[a_{n+1}, a_n)$, and hence the interval $[\frac{1}{2}\pi, \pi)$. Hence, this union contains an interval around 0, so that the resulting set $W = 2E \setminus E$ is a wavelet set. We may define T', arbitrarily but in accordance with the construction in Theorem 2.3, so that $T'(x) < 0$ for all x, implying that $E \cap [0, \infty) = T(Q) \cap [0, \infty) = \cup_{n=1}^\infty \cup_{j=n}^\infty \frac{1}{2^{j+1}}[a_{n+1}, a_n)$. One sees from this that W contains 0 as a limit point. Indeed, $W = 2E \setminus E$ contains all the intervals of the form $\frac{1}{2^n}[a_{n+1}, a_n)$.

(ii) Analogs of the above construction in higher dimensions give examples like the "hole in the middle"set of Soardi and Weiland ([SW]). Thus, taking $\{a_n\}$ as above, let T_n be the trapezoid in the first quadrant determined by the four points $(a_n, 0), (a_{n+1}, 0), (0, a_n)$, and $(0, a_{n+1})$, and let U_n and V_n be the analogous trapezoids in the second and fourth quadrants. Define $T(x) = \frac{x}{2}$ if

$$x \in \cup_{n=1}^\infty \cup_{j=n}^\infty \frac{1}{2^{j+1}}(T_n \cup U_n \cup V_n).$$

Define T on the rest of the cube Q in the first, second, and fourth quadrant s by $T(x) = \frac{x}{2} + (-\pi, -\pi)$, and define T on the third quadrant by $T(x) = \frac{x}{2}$. Now define T' in any way consistent with the construction in Theorem 2.3 so that $T'(x)$ is always in the third quadrant. As above, the union of the dilates of E contains a neighborhood of 0, so that we produce a wavelet set $W = 2E \setminus E$. Moreover, as above, the wavelet set W contains 0 as a limit point. Indeed, it contains a sequence of trapezoids converging to 0.

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