Constructing Wavelets from Generalized Filters

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Abstract

Over the past twenty years, wavelets have gained popularity as bases for transforms used in image and signal processing. We begin by showing how wavelets arise naturally in this context. Classical construction techniques using Fourier analysis are then presented. The paper concludes with recent extensions of these techniques employing the tools of abstract harmonic analysis and spectral multiplicity theory.

1 Introduction

Wavelets arise naturally in efforts to store images efficiently. To capture a black and white image on a 1600 by 1200 pixel computer screen we might first try storing a gray scale number between 0 and 255 for each of the 1,920,000 pixels. However, pixel by pixel storage is not very efficient, because it does not take advantage of regions in which the darkness does not change. For example, there are clearly more efficient ways to store an image of a black rectangle covering half of the screen, than to keep 960,000 copies of the number 0 and 960,000 copies of the number 255. Even a photograph of a face usually has large regions of constant darkness.

To overcome the inefficiency of pixel by pixel storage, we would like to use different levels of resolution in different regions of the image. In areas where darkness is highly variable, we need a higher level of resolution than in areas where it stays constant. As a first step toward this goal, we capture the whole image at different levels of resolution as follows: First we record the average gray scale on the whole image, which for convenience we think of as occupying the unit square. (For more general images, we can think of averaging over each of the 1×1 squares whose vertices are lattice points.) We call this the 0^{th} level of resolution. Then we record the average on each $\frac{1}{2} \times \frac{1}{2}$ subsquare, which we call the 1^{st} level of resolution. We can proceed to the resolution of single pixels by successively averaging our image and recording that average on each of the $\frac{1}{2^j} \times \frac{1}{2^j}$ subsquares (called the j^{th} level of resolution), for larger and larger j. This process yields a sequence of approximations to our image. We will have captured our image completely accurately at the j^{th} level if it was of constant darkness on all of the subsquares of a $\frac{1}{2^j} \times \frac{1}{2^j}$ grid.

Mathematically, we can describe this process in terms of a sequence of closed subspaces of $L^2(\mathbb{R}^2)$ given by V_j = functions constant on $\frac{1}{2^j} \times \frac{1}{2^j}$ squares. Our approximation at the j^{th} level of resolution is simply the closest L^2 approximation to our image in the subspace V_j. If we allow ourselves to both zoom in and zoom out arbitrarily far, i.e. to consider $-\infty < j < \infty$, we will have a structure of the following type, first defined by S. Mallat [13]:

Definition 1 *A* Multiresolution Analysis (MRA) *in $L^2(\mathbb{R}^n)$ is a collection of closed subspaces V_j that have the following properties:*

1. $V_j \subset V_{j+1}$
2. $V_{j+1} = \{\delta(f) \equiv 2f(2x)\}_{f \in V_j}$
3. $\cup V_j$ *is dense in* $L^2(\mathbb{R}^n)$ *and* $\cap V_j = \{0\}$
4. V_0 *has a* scaling function ϕ *whose translates form an orthonormal basis for* V_0

Property 2 explicitly defines a dilation operator on $L^2(\mathbb{R}^n)$ that takes us between different levels of resolution. The normalization factor of 2 makes this dilation a unitary operator. The first three properties together describe how the different levels of resolution are related in a way that reflects the successive capturing of our image. The final property describes how we can use a second unitary operator of translation to move around at the 0^{th} level (and thus at any fixed level if we conjugate by dilation). In our image example, ϕ is the characteristic function of the unit square.

By using an MRA, we have achieved our preliminary goal of capturing our image at different levels of resolution. However, we have not yet gained efficiency over pixel by pixel storage unless our image is, like the rectangle, an element of one of the V_j spaces. Indeed, if we continue our process down to the level of pixel by pixel resolution, we will have all the inefficiency we started with, together with information from all the previous levels of resolution as well. The problem is that we are starting over at each level, so that there is redundancy in the information stored at successive levels. To see an explicit example of this, notice that in going from the 0^{th} level to the 1^{st}, we already know the overall average gray scale value, and thus would only need to record the averages on three of the four subsquares to have total information about all four subsquare averages.

To overcome the redundancy, instead of storing all of the V_1 information in addition to V_0's, we write $V_1 = V_0 \oplus W_0$ and seek an orthonormal basis for W_0. In our example, we let q_1, q_2, q_3, and q_4 be the upper left, upper right, lower left, and lower right quadrants of the unit square respectively, and let

$$\psi_1 = \chi_{q_1 \cup q_2} - \chi_{q_3 \cup q_4},$$

$$\psi_2 = \chi_{q_1 \cup q_3} - \chi_{q_2 \cup q_4}$$

and

$$\psi_3 = \chi_{q_1 \cup q_4} - \chi_{q_2 \cup q_3},$$

where χ_A denotes the characteristic function of the set A. Then the translates of ψ_1, ψ_2, ψ_3 and ϕ form an orthornormal basis for V_1. In fact, positive and negative dilates of translates of just ψ_1, ψ_2 and ψ_3 form an orthonormal basis for $L^2(\mathbb{R}^2)$. Storing the coefficients of our image in terms of its coefficients for the orthonormal basis given by the dilates and translates of the ψ's does finally achieve the image compression we were seeking. At each level, the new information given by the coefficients of the further dilated ψ's can be thought of as correction terms to update the information from the previous level of resolution. In regions of the image where darkness does not change, these coefficients will all eventually be 0. Thus we achieve *lossless* compression from the savings of storing sequences containing lots of zeroes. We can accomplish further compression with the least loss of accuracy in the image by throwing away the coefficients that are smallest in absolute value. Since our dilation operator normalizes at each step, the coefficients will give an accurate measure of the relative importance of the correction terms.

The ψ's are called a wavelet. In general we have:

Definition 2 *$\{\psi_k\}_{k=1..r} \subset L^2(\mathbb{R}^n)$ is an* orthonormal wavelet *for dilation by an integral expansive matrix D if $\{\psi_{j,k,l} \equiv \sqrt{|\det D|}^j \psi_k(D^j x - l)\}_{j,l \in \mathbb{Z}; k=1..r}$ form an orthonormal basis for $L^2(\mathbb{R}^n)$.*

We began our description of wavelets in $L^2(\mathbb{R}^2)$ in order to show their relationship to the problem of image compression, but the simplest place to study wavelets is in 1-dimension. Three well-known and simple examples of wavelets for dilation by 2 in $L^2(\mathbb{R})$ are the Haar wavelet [11],

$$\psi = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)},$$

the Shannon wavelet, for which

$$\widehat{\psi} = \chi_{[-1,-\frac{1}{2}) \cup [\frac{1}{2},1)}$$

and the Journé wavelet [13], with

$$\widehat{\psi} = \chi_{[-\frac{16}{7},-2) \cup [-\frac{1}{2},-\frac{2}{7}) \cup [\frac{2}{7},\frac{1}{2}] \cup [2,\frac{16}{7})}.$$

We can think of the 2-dimensional example we developed above as being built out of tensor products of the 1-dimensional Haar wavelet. The other two 1-dimensional examples given here are described in terms of their Fourier transform $\widehat{\psi}$. They are interesting as wavelets because of the simplicity of these transforms. In particular, the Shannon wavelet is so-named because of its relationship to the Shannon sampling formula [15]. Note that on the Fourier transform side, translation becomes multiplication by exponentials. We will see in the next section that this fact makes the Fourier transform very useful in the study of wavelets.

Two basic questions arise in looking at the examples given above:

1. How do we find new wavelets with desirable properties? The examples of wavelets we have described so far all have their drawbacks. The Haar wavelet in either dimension 1 or 2 is discontinuous, and thus is not the best basis to use to capture smooth images. The Shannon and Journé wavelets have Fourier transforms that are discontinuous, and thus are not well localized. Can we find a smooth wavelet for dilation by 2 in $L^2(\mathbb{R})$ that still has compact support? Another natural question about finding new wavelets concerns the number of ψ_k's required. The fact that our examples so far consist of single wavelets ($r = 1$ in Definition 2) in 1 dimension, but a 3-wavelet in 2 dimensions also raises the question of whether we can find a 1-wavelet or 2-wavelet for dilation by 2 in $L^2(\mathbb{R}^2)$.

2. How strong is the connection between wavelets and MRA's? Although we motivated the definition of wavelet using the idea of an MRA, it turns out that some wavelets (for example, the Journé wavelet above) have no associated MRA's.

The first question is easiest to answer if we assume we have an MRA for dilation by 2 in $L^2(\mathbb{R})$. This is the setting in which Meyer [14] and Daubechies [10] carried out their famous construction of wavelets using filters. We describe that work in Section 2 below. Their answer can then be generalized to situations where no MRA is possible. This leads to the work of Baggett, Courter, Jorgensen, Medina, Packer and Merrill, which is described in Section 3.

2 Building MRA wavelets from filters in $L^2(\mathbb{R})$

Suppose we have a single wavelet ψ for dilation by 2 in $L^2(\mathbb{R})$, with an associated MRA, and so a scaling function ϕ whose translates form an orthonormal basis for V_0.

Because $\widehat{V_0} \subset \widehat{V_1}$, and $\widehat{W_0} \subset \widehat{V_1}$, we can write $\widehat{\phi} \in \widehat{V_0}$ and $\widehat{\psi} \in \widehat{W_0}$ in terms of exponentials times the dilate of $\widehat{\phi}$. That is, there must exist periodic functions (with period 1) h and g such that

$$\widehat{\phi}(x) = \frac{1}{\sqrt{2}} h(\frac{x}{2}) \widehat{\phi}(\frac{x}{2}) \tag{1}$$

and

$$\widehat{\psi}(x) = \frac{1}{\sqrt{2}} g(\frac{x}{2}) \widehat{\phi}(\frac{x}{2}). \tag{2}$$

EXAMPLES: For the Shannon wavelet, where $\widehat{\psi} = \chi_{[-1,-\frac{1}{2}) \cup [\frac{1}{2},1)}$ and $\widehat{\phi} = \chi_{[-\frac{1}{2},\frac{1}{2})}$, we have

$$h = \sqrt{2}\chi_{[-\frac{1}{4},\frac{1}{4})} \text{ and } g = \sqrt{2}\chi_{[-\frac{1}{2},-\frac{1}{4}) \cup [\frac{1}{4},\frac{1}{2})}.$$

For the Haar wavelet, where $\phi = \chi_{[0,1)}$ and $\psi = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}$, we have

$$h = \frac{1}{\sqrt{2}}(1 + e^{2\pi i x}) \text{ and } g = \frac{1}{\sqrt{2}}(e^{2\pi i x} - 1).$$

The functions h and g are called **low and high pass filters**. Notice that in the Shannon example in particular, h and g do indeed act by filtering out all but low (for h) or high (for g) frequencies. Because of the orthonormality conditions satisfied by translates of ϕ and ψ, all filters defined by (1) and (2) must satisfy orthonormality-like conditions:

$$|h(x)|^2 + |h(x+\frac{1}{2})|^2 = 2 \tag{3}$$

$$|g(x)|^2 + |g(x+\frac{1}{2})|^2 = 2 \tag{4}$$

and

$$h(x)\overline{g(x)} + h(x+\frac{1}{2})\overline{g(x+\frac{1}{2})} = 0. \tag{5}$$

The reason filters are useful is that we can reverse this process of finding filters from wavelets. First note that we can easily build a high pass filter to go with any low pass filter. Indeed, if h is any periodic function that satisfies (3), we can take

$$g(x) = e^{2\pi i x}\overline{h(x+\frac{1}{2})},$$

and the other two orthonormality conditions (4) and (5) will be satisfied as well. (Other choices for g are possible.) Given h and g, under appropriate conditions, we can then build $\widehat{\phi}$ by iterating equation (1). The appropriate conditions are exactly those that are needed to make the resulting infinite product converge.

Theorem 1 *Let h and g be C^1 functions that satisfy the orthonormality conditions (3), (4), and (5). Suppose, in addition, that h is nonvanishing on $[-\frac{1}{4}, \frac{1}{4})$, and $|h(0)| = \sqrt{2}$. Then:*

$$\widehat{\phi}(x) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{2}} h(2^{-j}x)$$

is a scaling function for an MRA, and

$$\widehat{\psi}(x) = \frac{1}{\sqrt{2}} g(\frac{x}{2})\widehat{\phi}(\frac{x}{2}).$$

is an orthonormal wavelet.

This technique was developed by Mallat [13] and Meyer [14], and used by Daubechies [10] to build C^r wavelets with compact support. (It can be shown that there are no C^∞ wavelets with compact support.) Daubechies' construction uses powers of the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$, which fits naturally into equation 3 to find the lowpass filter h. A good introductory description of these constructions appears in [16].

The theorem's requirements that h satisfy $|h(0)| = \sqrt{2}$ and h be in C^1 are natural restrictions in order to make the infinite product converge. The condition that h is nonvanishing on $[-\frac{1}{4}, \frac{1}{4}]$ appears less natural; it is used in the proof to ensure L^2 convergence of the infinite product and thus the orthonormality of the translates of ϕ. A famous example due to A. Cohen [6] showed that removing the condition h nonvanishing on $[-\frac{1}{4}, \frac{1}{4})$ can in fact lead to functions ϕ and ψ whose translates are not orthonormal. Cohen took $h = \frac{1+e^{-6\pi i x}}{\sqrt{2}}$, which resulted in a stretched out version of the Haar scaling function and wavelet, $\phi = \frac{1}{3}\chi_{[0,3)}$ and $\psi = \frac{1}{3}(\chi_{[-\frac{1}{2},1)} - \chi_{[1,\frac{5}{2})})$.

However, the classical Theorem 1 can be extended to accommodate this and similar examples if we generalize our definition of wavelet.

Definition 3 *$\{\psi_{j,k,l}\}$ is a* normalized tight frame *for $L^2(\mathbb{R}^n)$ if for each $f \in L^2$ we have $\|f\|^2 = \sum_{j,k,l} |\langle f|\psi_{j,k,l}\rangle|^2$.*

$\{\psi_k\} \subset L^2(\mathbb{R}^n)$ is a frame wavelet *for dilation by an integral expansive matrix D if $\{\psi_{j,k,l} \equiv \sqrt{|\det D|}^j \psi_k(D^j x - l)\}$ form a normalized tight frame for $L^2(\mathbb{R}^n)$.*

Note that a normalized tight frame can exhibit redundancy, and therefore need not be a basis. Indeed, it can include 0 as one of its elements. However, a normalized tight frame $\{f_j\}$ does have the property that every $f \in L^2$ can be recaptured from its coefficients, $f = \sum \langle f, f_j\rangle f_j$. It turns out that a normalized tight frame of unit vectors must be an orthonormal basis. (See, e.g. [**?**]).

By broadening our definition of wavelet to include frame wavelets, we get the following generalization of Theorem 1, which appears in [5] and was proven independently in [12] for $d = 2$:

Theorem 2 *Suppose $h, g_1, \cdots g_{d-1}$ are periodic Lipschitz continuous function in $L^2(\mathbb{R})$, which satisfy $|h(0)| = \sqrt{d}$ and the filter equations*

1. $\sum_{l=0}^{d-1} |h(x + \frac{l}{d})|^2 = d$
2. $\sum_{l=0}^{d-1} h(x + \frac{l}{d})\overline{g_i(x + \frac{l}{d})} = 0$
3. $\sum_{l=0}^{d-1} g_i(x + \frac{l}{d})\overline{g_j(x + \frac{l}{d})} = d\delta_{i,j}$

then the construction $\widehat{\phi}(x) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{d}} h(d^{-j}x)$ produces an L^2 function ϕ (whose translates are not necessarily orthogonal), and the $d-1$ functions $\widehat{\psi_k}(x) = \frac{1}{\sqrt{d}} g_k(\frac{x}{d})\widehat{\phi}(\frac{x}{d})$ form a frame wavelet for dilation by d in $L^2(\mathbb{R})$.

This extension starts with a filter from an MRA orthonormal wavelet and produces a frame wavelet that need not be associated with an MRA. Thus it suggests that we look again at the connection between wavelets and MRA's, and consider the possibility of more general filters.

3 Generalized Multi-resolution Analyses and Generalized Filters

As mentioned in the introduction, even orthonormal wavelets need not be associated with an MRA. The reason for this lies in the MRA requirement of the existence of a scaling function. Given an orthonormal wavelet $\{\psi_k\} \subset L^2(\mathbb{R}^n)$, if we let $V_j =$ the closed linear span of $\{\psi_{i,k,l}\}_{i<j}$, then the subspaces $\{V_j\}$ do determine a *generalized* MRA, according to the definition below:

Definition 4 *A* Generalized Multiresolution Analysis (GMRA) *is a collection of closed subspaces* $\{V_j\}_{j\in\mathbb{Z}}$ *of* $L^2(\mathbb{R}^n)$ *such that:*

1. $V_j \subset V_{j+1}$
2. $V_{j+1} = \{\delta_D(f) \equiv \sqrt{|\det D|} f(Dx)\}_{f\in V_j}$
3. $\cup V_j$ *dense in* $L^2(\mathbb{R}^n)$ *and* $\cap V_j = \{0\}$
4. V_0 *is invariant under translation.*

The definitions of MRA and GMRA differ only in condition (4): An MRA requires that V_0 has a scaling function ϕ such that translates of ϕ form an orthonormal basis for V_0, while a GMRA requires only that V_0 be invariant under translation by the integer lattice. In spite of this difference, it is shown in [3] that a GMRA has almost as much structure as an MRA. Translation is a unitary representation of $\mathbb{Z}^n$ on V_0, and thus is completely determined by a multiplicity function $m : [-\frac{1}{2}, \frac{1}{2})^n \mapsto \{0, 1, 2, \cdots, \infty\}$ describing how many times each character occurs as a subrepresentation. A GMRA is an MRA iff $m \equiv 1$. Journé's famous non-MRA wavelet example for dilation by 2 in $L^2(\mathbb{R})$ has

$$m(x) = \begin{cases} 2 & x \in [-\frac{1}{7}, \frac{1}{7}) \\ 1 & x \in \pm[\frac{1}{7}, \frac{2}{7}) \cup \pm[\frac{3}{7}, \frac{1}{2}) \\ 0 & \text{otherwise} \end{cases}$$

In any GMRA, we write $V_1 = V_0 \oplus W_0$, just as we did in the MRA case. Representation theory can then be used (see [3]) to show that the GMRA has an associated orthonormal wavelet if and only if the multiplicity function satisfies a *consistency equation*:

$$m(x) + (\text{number of wavelets}) = \sum m(\text{preimages of } x \text{ under } D^t).$$

Taking $m \equiv 1$, this consistency equation determines that the number of wavelets must be 3 for an MRA wavelet for dilation by 2 in $L^2(\mathbb{R}^2)$, so 1-wavelets cannot be found there using the MRA filter technique. However, examples of non-MRA, GMRA orthonormal wavelets whose Fourier transforms are characteristic functions can be built directly from the consistency equation. We have used this technique (in [3]and [1]) to make 1-wavelets and a 2-wavelet in $L^2(\mathbb{R}^2)$, and a

4-wavelet in $L^2(\mathbb{R}^3)$. (By the consistency equation, MRA wavelets require 7-wavelets in $L^2(\mathbb{R}^3)$.) Other examples of non-MRA 1-wavelets in $L^2(\mathbb{R}^2)$ appear in, e.g. [9] and [4].

In every GMRA, there is a unitary equivalence between translation on V_0 and multiplication by exponentials on $\oplus L^2(S_j)$, where $S_j = \{x : m(x) \geq j\}$. This unitary equivalence plays a role here similar to that of the Fourier transform in the classical MRA case. It ensures that in a GMRA we can find *generalized scaling functions*, $\phi_1, \phi_2, \cdots$ such that $\{\phi_i(x-l)\}$ form a normalized tight frame for V_0. It also enables us to develop a generalized notion of low and high pass filters.

Just as in the classical case, we begin by building filters from wavelets, then see if we can reverse the process. Suppose we have a non-MRA orthonormal k-wavelet $\psi_1, \cdots, \psi_k$ for dilation by D in $L^2(\mathbb{R}^n)$. We then have generalized scaling functions $\phi_1, \cdots, \phi_c$. Since $V_0 \subset V_1$ and $W_0 \subset V_1$, we can show ([1]) there exist periodic functions $h_{i,j}$ and $g_{l,j}$, supported on the periodization of S_j, such that

$$\widehat{\phi_i}(x) = \frac{1}{\sqrt{|\det D|}} \sum_{j=1}^{c} h_{i,j}((D^t)^{-1}x)\widehat{\phi_j}((D^t)^{-1}x) \tag{6}$$

and

$$\widehat{\psi_l}(x) = \frac{1}{\sqrt{|\det D|}} \sum_{j=1}^{c} g_{l,j}((D^t)^{-1}x)\widehat{\phi_j}((D^t)^{-1}x). \tag{7}$$

These *generalized filters* $g_{i,j}$ and $h_{i,j}$ satisfy orthonormality-like conditions that are generalizations of the classical conditions (3),(4) and (5):

$$\sum_{j=1}^{c} \sum_{l=0}^{|\det D|-1} h_{i,j}(x_l)\overline{h_{k,j}(x_l)} = (\det D)\delta_{i,k}\chi_{S_i}(x), \tag{8}$$

$$\sum_{j=1}^{c} \sum_{l=0}^{|\det D|-1} g_{i,j}(x_l)\overline{g_{k,j}(x_l)} = (\det D)\delta_{i,k}, \tag{9}$$

and

$$\sum_{j=1}^{c} \sum_{l=0}^{|\det D|-1} h_{i,j}(x_l)\overline{g_{k,j}(x_l)} = 0, \tag{10}$$

where the x_l are the preimages of x under D^t mod 1.

In the case of the Journé wavelet, equations (6) and 7 simplify to:

$$\widehat{\phi_1}(x) = \frac{1}{\sqrt{2}}\left(h_{1,1}((\frac{x}{2})\widehat{\phi_1}(\frac{x}{2}) + h_{1,2}((\frac{x}{2})\widehat{\phi_2}(\frac{x}{2})\right)$$

$$\widehat{\phi_2}(x) = \frac{1}{\sqrt{2}}\left(h_{2,1}((\frac{x}{2})\widehat{\phi_1}(\frac{x}{2}) + h_{2,2}((\frac{x}{2})\widehat{\phi_2}(\frac{x}{2})\right),$$

and

$$\widehat{\psi}(x) = \frac{1}{\sqrt{2}}\left(g_1((\frac{x}{2})\widehat{\phi_1}(\frac{x}{2}) + g_2((\frac{x}{2})\widehat{\phi_2}(\frac{x}{2})\right),$$

which look very similar to the equations (1) and (2) that define classical low and high pass filters. We can use these to find the Journé generalized filters from the wavelet

$$\widehat{\psi} = \chi_{[-\frac{16}{7},-2)\cup[-\frac{1}{2},-\frac{2}{7})\cup[\frac{2}{7},\frac{1}{2}]\cup[2,\frac{16}{7})},$$

and generalized scaling functions

$$\widehat{\phi_1}(x) = \chi_{[-\frac{4}{7},-\frac{1}{2})\cup[-\frac{2}{7},\frac{2}{7})\cup[\frac{1}{2},\frac{4}{7})},\ \widehat{\phi_2}(x) = \chi_{[-\frac{8}{7},-1)\cup[1,\frac{8}{7})}.$$

We obtain (see [7])

$$h_{1,1} = \sqrt{2}\chi_{[-\frac{2}{7},-\frac{1}{4})\cup(-\frac{1}{7},\frac{1}{7})\cup[\frac{1}{4},\frac{2}{7})}\ ,\ h_{1,2} = 0,$$

$$h_{2,1} = \sqrt{2}\chi_{[-\frac{4}{7},-\frac{1}{2})\cup[\frac{1}{2},\frac{4}{7})}\ ,\ h_{2,2} = 0$$

$$g_1 = \sqrt{2}\chi_{[-\frac{1}{4},-\frac{1}{7})\cup[\frac{1}{7},\frac{1}{4})}\ , \text{ and } g_2 = \sqrt{2}\chi_{[-\frac{1}{7},\frac{1}{7})}.$$

To use generalized filters to build wavelets, we now wish to reverse this procedure, just as we did in the classical case. In order to first build filters, we use functions on the disjoint union of the S_j's whose values are $\sqrt{\det D}$ times unitary matrices, with different dimensions for different values of x. We need the values of the filters to be $\sqrt{\det D}$ times unitary matrices in order to satisfy the generalized orthonormality conditions (8), (9), and (10). The matrices of filter values have different dimensions depending on how many of the sets S_j the point x and its preimages are in. Once we have the filters, we build the generalized scaling function using an infinite product of matrices that comes from the iteration of equation(6). The wavelet is then produced by equation(7). Conditions that make this possible are described in the following generalization (see [2]) of the Bratteli-Jorgensen theorem :

Theorem 3 *Suppose $\{h_{i,j}\}$ and $\{g_{k,j}\}$ are periodic functions that are supported on the periodization of S_j, Lipschitz continuous in a neighborhood of the origin, and that satisfy the three generalized orthonormality conditions (8),(9), and (10). Suppose in addition the $h_{i,j}$ satisfy the generalized lowpass conditions $h_{i,j} = 0$ for $j > i$ and $|h_{i,j}(0)| = \sqrt{(|\det D|)}\delta_{(i,1)}\delta_{(j,1)}$. Write H for the matrix $(h_{i,j})$. Then the components of $\prod_{k=1}^{\infty} \frac{1}{\sqrt{|\det D|}} H((D^t)^{-k}x)$ converge pointwise to $P_{i,j} \in L^2(\mathbb{R}^n)$. If we let $\widehat{\phi_i} = P_{i,1}$, then*

$$\widehat{\psi_k}(x) \equiv \frac{1}{\sqrt{|\det D|}} \sum_j g_{k,j}((D^t)^{-1}x)\widehat{\phi_j}((D^t)^{-1}x)$$

are the Fourier transforms of a frame wavelet on $L^2(\mathbb{R}^n)$.

The proof, like that of [5], proceeds by using matrices of values of the filters to define partial isometries that satisfy relations similar to those defining a Cuntz algebra [8].

Using this procedure, we have built wavelets with interesting properties, for example, a non-MRA orthonormal wavelet on $L^2(\mathbb{R})$ whose Fourier transform is C^∞ on an arbitrarily large interval (see [1]), and a non-MRA frame wavelet on $L^2(\mathbb{R})$ whose Fourier transform is C^∞ on all of $\mathbb{R}$ (see [2]). These examples are somewhat surprising since it is known that compactly supported wavelets on $\mathbb{R}$ must be MRA wavelets.

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