KEY

Calculus 3–Block 8, 2016

Midterm Exam

Take the midterm in Tutt Science Center or the Fishbowl. You may not use books, calculators, computers or notes (except for the one 8-1/2"x11" page you prepared for yourself in advance). Show all your work. Do not discuss the exam with anyone except me. Sign the Honor Code when you are finished, and bring the exam, along with your 8-1/2"x11" page of notes to my office (TSC 206F). The exam is due by 3PM.

1. Determine whether each of the following series converges or diverges and give a convincing reason why. Where appropriate, distinguish between absolute and conditional convergence.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}(1+\sqrt{n})} = \sum \frac{1}{n^{1/3}+n^{5/6}}$ Limit compare to $\frac{1}{n^{5/6}}$

3 pts

$\lim_{n\to\infty} \frac{n^{5/6}}{n^{1/3}+n^{5/6}} = \lim_{n\to\infty} \frac{1}{n^{-1/2}+1} = 1$ so behave the same

Diverges $p = 5/6 < 1$

b. $\sum_{n=1}^{\infty} \frac{(2n)!\,3^n}{(3n)!\,2^n}$ Ratio test $\lim_{n\to\infty} \frac{(2n+2)!\,3^{n+1}\cdot(3n)!\,2^n}{(3n+3)!\,2^{n+1}(2n)!\,3^n}$

3 pts

$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)3}{(3n+3)(3n+2)(3n+1)2} \cdot \frac{1/n^3}{1/n^3} = \frac{(2+2/n)(2+\frac{1}{n})\frac{3}{n}}{(3+3/n)(3+2/n)(3+\frac{1}{n})2} \to 0 < 1$

Converges

c. $\sum_{n=1}^{\infty}(-1)^n(\frac{n}{10n-3})^2$

$(-1)^n\left(\frac{n}{10n-3}\right)^2$ alternates between numbers close to $\frac{1}{100}$ and numbers close to $\frac{-1}{100}$ so does not approach 0. Diverges by n^{th} term test.

3 pts

2. Find the Taylor series for $f(x) = x^{100}$ centered around $x = 1$.

5 pts

$F(1) = 1$

$F'(1) = 100$

$F''(1) = 100 \cdot 99$

$f'''(1) = 100 \cdot 99 \cdot 98$

$\vdots$

$F^{(100)}(1) = 100!$

$f^{(101)}(1) = 0.$

$\vdots$

$$f(x) = \sum_{n=0}^{100} \frac{100!}{n!\,(100-n)!}(x-1)^n$$

$$= \sum_{n=0}^{100} \binom{100}{n}(x-1)^n$$

3. Find the interval of convergence of $\sum_{n=2}^{\infty} \frac{(x-3)^n}{n \ln n}$

5 pts

Ratio Test

$$\lim_{n\to\infty} \frac{|x-3|^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{|x-3|^n} = |x-3| \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)}$$

$$= |x-3| \cdot 1 \cdot \lim_{n\to\infty} \frac{1/n}{1/n+1} \quad \text{(L'Hospital)}$$

$$= |x-3| \quad \text{so conv abs } |x-3|<1 \quad \text{div. } |x-3|>1$$

check endpoints. $x=2$ and $x=4$

$x=4$ $\sum \frac{1}{n \ln n}$ Integral test $\int_0^b \frac{1}{x \ln x} = \ln(\ln x)\Big]_0^b \to \infty$ as $b \to \infty$

so Diverges

$x=2$ $\sum \frac{(-1)^n}{n \ln n}$ converges by alt series test

So interval is $[2, 4)$

4. Let $\vec{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$ be the position vector of a particle travelling in space.

a. Find the speed of the particle at time t.

3 pt

$r'(t) = (-e^t \sin t + e^t \cos t,\ e^t \cos t + e^t \sin t,\ e^t)$

Speed $= \|r'(t)\| = \sqrt{e^{2t}\sin^2 t - 2e^{2t}\sin t\cos t + e^{2t}\cos^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t\cos t + e^{2t}\sin^2 t + e^{2t}}$

$= \sqrt{3e^{2t}} = \boxed{\sqrt{3}\, e^t}$

b. How long from the time $t = 0$ does it take the particle to travel a distance 3 along its path?

3 pt

$s = \int_0^T \sqrt{3}\, e^t\, dt = \sqrt{3} \int_0^T e^t dt = \sqrt{3}\,(e^T - 1)$

we want $s = 3$ so $3 = \sqrt{3}\, e^T - 1$

$\sqrt{3} + 1 = e^T$

$\boxed{\ln(\sqrt{3}+1) = T}$

c. Find the curvature of the particle's path at time $t = 0$. (You do not need to find a formula for curvature at an arbitrary point.)

3 pt

$T = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{3}}(-\sin t + \cos t,\ \cos t + \sin t,\ 1)$

so $\frac{dT}{dt} = \frac{1}{\sqrt{3}}(-\cos t - \sin t,\ -\sin t + \cos t,\ 0)$

so $K = \frac{\|\frac{dT}{dt}\|}{\frac{ds}{dt}} = \frac{\frac{1}{\sqrt{3}}\sqrt{2}}{\sqrt{3}} = \boxed{\frac{\sqrt{2}}{3}}$ at $t = 0$

OR

$r'' = (e^t(-\cos t - \sin t) + e^t(-\sin t + \cos t),$
$e^t(-\sin t + \cos t) + e^t(\cos t + \sin t), e^t)$

so $r''(0) = (0, 2, 1)$

$r''(0) \times r'(0) = \begin{vmatrix} i & j & k \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i} + \vec{j} - 2\vec{k}$

so $K = \frac{\|r'' \times r'\|}{\|r'\|^3} = \frac{\sqrt{6}}{3\sqrt{3}} = \boxed{\frac{\sqrt{2}}{3}}$

d. Find the tangential and normal components of acceleration at time $t = 0$.

2 pt

tangential: $\frac{dv}{dt} = \sqrt{3}\, e^t$ at $t = 0$ $= \boxed{\sqrt{3}}$

normal. $Kv^2 = \frac{\sqrt{2}}{3}(\sqrt{3})^2 = \boxed{\sqrt{2}}$

5. $f(x, y) = x^3 - 6xy + y^2$

a. Find all local minima, local maxima, and saddle points of $z = f(x, y)$ (tell which is which).

$F_x = 3x^2 - 6y = 0$ when $y = \frac{x^2}{2}$ so critical points $3x = \frac{x^2}{2}$

$f_y = -6x + 2y = 0$ when $y = 3x$ $x^2 - 6x = 0$ $x = 0$ or $x = 6$

$F_{xx} = 6x$ $F_{xy} = -6$ $f_{yy} = 2$ so critical points $(0,0)$ and $(6,18)$

$D = 12x - 36$ at $x = 0$ $D < 0$ saddle

at $x = 6$ $D > 0$ $F_{xx} > 0$ local min.

$\boxed{\text{local min at } (6,18) \quad \text{saddle at } (0,0)}$

b. Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point (1,0).

$(F_x, f_y, -1) \cdot (x-1, y, z-1) = 0.$ (at 1,0), $F_x = 3$ and $f_y = -6$

$3(x-1) - 6y - 1(z-1) = 0$

$\boxed{3x - 6y - z - 2 = 0}$

c. Find the equation of a line (not just a vector) normal to the surface at this point.

normal vector from (b) is $(3, -6, -1)$

point is point of tangency $(1, 0, 1)$

so $\boxed{r(t) = (1,0,1) + (3,-6,-1)t.}$