45 points.

KEY

the identities
$\cos^2\theta = \frac{1+\cos 2\theta}{2}$
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$
might help you.

Calculus 3–Block 8, 2016

## Final Exam

Take the midterm in Tutt Science Center or the Fishbowl. You may not use books, computers or notes except for the one 8-1/2"x11" page you prepared for yourself in advance. You may use a calculator for arithmetic and to evaluate trig and exponential functions only–no graphing, symbolic algebra, calculus or programmable functions. Show all your work. Do not discuss the exam with anyone except me. Sign the Honor Code when you are finished, and bring the exam, along with your 8-1/2"x11" page of notes to my office (TSC 206F). The exam is due by 12PM.

6 pts

1. Find the Taylor series centered at 0 for $\frac{1}{4x-1}$, and then find the interval on which it converges.

$f(x) = (4x-1)^{-1}$ $\quad f(0) = -1 \quad a_0 = -1$

$f'(x) = -(4x-1)^{-2}(4)$ $\quad f'(0) = -4 \quad a_1 = -4$

$f''(x) = 2(4x-1)^{-3}(4)^2$ $\quad f''(0) = -4^2 \cdot 2 \quad a_2 = -4^2$

$f'''(x) = -6(4x-1)^{-4}(4)^3$ $\quad f'''(0) = -4^3 \cdot 3! \quad a_3 = -4^3$

$\vdots$

$f^{(n)}(x) = (-1)^n n!(4x-1)^{-n-1}(4)^n$ $\quad f^{(n)}(0) = -4^n \cdot n! \quad a_n = -4^n$

$$\frac{1}{4x-1} = \boxed{-\sum_{n=0}^{\infty} 4^n x^n}$$

Radius of convergence: Ratio test $\lim_{n\to\infty} \frac{4^{n+1}|x|^{n+1}}{4^n|x|^n} = 4|x| < 1$

so $|x| < \frac{1}{4}$ converges absolutely $\quad |x| > \frac{1}{4}$ diverges.

Check endpoints

at $x = \frac{1}{4}$ $\quad \sum_{n=0}^{\infty} 1$ diverges

at $x = \frac{-1}{4}$ $\quad \sum_{n=0}^{\infty} (-1)^n$ diverges

so converges on $\boxed{\left(\frac{-1}{4}, \frac{1}{4}\right)}$

2. Suppose $\vec{F}(x,y) = x^2y\vec{i} + y\vec{j}$ and $\vec{G}(x,y) = 2xy\vec{i} + x^2\vec{j}$. Let $C_1$ be the straight line segment from (0,1) to (1,1) and $C_2$ be the ccw oriented circle of radius 1 centered at the origin. Consider the four integrals: $\int_{C_1} \vec{F}\cdot d\vec{r}$, $\int_{C_2} \vec{F}\cdot d\vec{r}$, $\int_{C_1} \vec{G}\cdot d\vec{r}$, and $\int_{C_2} \vec{G}\cdot d\vec{r}$.

3 pts

a. You can immediately tell that one of these integrals must be 0. Which one and why? $\vec{G}$ is conservative since $\frac{\partial G_1}{\partial y} = \frac{\partial G_2}{\partial x} = 2x$

$C_1$ is a closed curve.

So $\boxed{\int_{C_2} \vec{G}\cdot d\vec{r} = 0}$

3 pts

b. A second one of these integrals can also be evaluated without doing any integration by using the Fundamental Theorem of line integrals. Evaluate it this way.

Since $\vec{G}$ is conservative, $\vec{G} = \nabla g$ where $g(x,y) = x^2y + c_1(y)$
$= x^2y + c_2(x)$
$= x^2y$

so $\boxed{\int_{C_1} \vec{G}\cdot dr = g(1,1) - g(0,1) = 1}$

4 pts

c. A third one of these integrals can be calculated by using a double integral of a simpler (nonzero) function. Evaluate it this way.

By Green's Theorem $\boxed{\int_{C_2} \vec{F}\cdot dr} = \iint \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$

$= \iint_D -x^2 dA$ where $D$ is inside circle of rad 1

$= -\int_0^{2\pi}\int_0^1 r^3 \cos^2\theta \, dr\, d\theta$

$= -\int_0^{2\pi} \frac{r^4}{4}\Big]_0^1 \cos^2\theta\, d\theta$

$= -\frac{1}{4}\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta = -\frac{1}{4}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big]_0^{2\pi} = \boxed{\frac{-\pi}{4}}$

4 pts

d. Evaluate the integral you have not yet done by parameterizing the curve and calculating the line integral.

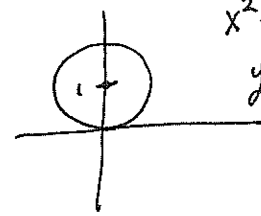
For $C_1$, $\vec{r}(t) = (0,1) + t(1,0) = (t,1)$
$\vec{r}'(t) = (1,0)$

$\boxed{\int_{C_1} \vec{F}\cdot d\vec{r}} = \int_0^1 (t^2, 1)\cdot(1,0)\, dt = \int_0^1 t^2 dt = \boxed{\frac{1}{3}}$

3 pts

3. a. Write down (but do not evaluate) a triple integral in rectangular coordinates that equals the volume that is between the surface $z = x^2 + y^2$ and the plane $z = 2y$. (Hint: to find the intersection between the surfaces, either complete the square or use polar coordinates.)

$$\int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \int_{x^2+y^2}^{2y} 1 \, dz\,dy\,dx$$




$x^2+y^2=2y$

$x^2+y^2-2y+1=1$

$y=1\pm\sqrt{1-x^2}$

3 pts

b. Rewrite your integral in (a) in cylindrical coordinates to give the same volume.

$$\int_0^{\pi} \int_0^{2\sin\theta} \int_{r^2}^{2r\sin\theta} 1 \, dz\, r\,dr\,d\theta$$

$x^2+y^2=2y$

$r^2=2r\sin\theta$

$r=2\sin\theta.$

3 pts

c. Rewrite your answer to either (a) or (b) as a double integral, and interpret this double integral as giving the mass of a 2-dimensional object. What region of the plane does that object occupy, and what is its density function?

$$\int_0^{\pi} \int_0^{2\sin\theta} 2r\sin\theta - r^2 \cdot r\,dr\,d\theta \qquad \delta = 2r\sin\theta - r^2$$

$$\int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} 2y - x^2 - y^2 \, dy\,dx \qquad \delta = 2y - x^2 - y^2$$

object is inside circle $x^2+(y-1)^2=1$

or in polar $r=2\sin\theta.$

5 pts

4. Find the value of $I = \int_{-\infty}^{\infty} e^{-x^2} dx$ (the bell curve important in probability) as follows: Write $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy$, then evaluate using polar coordinates.

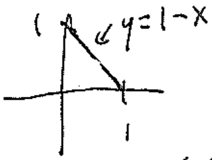
$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2} e^{-r^2} \Big]_0^{\infty} d\theta$$

$$= \frac{1}{2} \cdot 2\pi = \pi \qquad \text{so } \boxed{I = \sqrt{\pi}}$$

5. a. Find the flux of the vector field $\vec{F}(x,y,z) = xze^y\vec{i} - xze^y\vec{j} + z\vec{k}$ through the part of the plane $x+y+z=1$ in the first octant, with downward normal.

5pts




Plane can be described as $G(x,y) = (x, y, 1-x-y)$

$\vec{F}(G(x,y)) = \left((1-x-y)xe^y, -(1-x-y)xe^y, 1-x-y\right)$

$T_x = (1, 0, -1)$

$T_y = (0, 1, -1)$

$N = \pm T_x \times T_y = \pm(1, 1, 1) = (-1, -1, -1)$

$\iint_D \vec{F}\cdot\vec{N}\, dA$

$= -\int_0^1 \int_0^{1-x} 1-x-y \, dy\, dx$

$= -\int_0^1 y - xy - \frac{y^2}{2}\Big]_0^{1-x} = -\int_0^1 1-x-x+x^2 - \frac{1-2x+x^2}{2} dx = -\int_0^1 \frac{1}{2} - x - \frac{x^2}{2} dx = -\left(\frac{1}{2}x - \frac{x^2}{2} - \frac{x^3}{6}\right)\Big]_0^1 = \boxed{\frac{1}{6}}$

2pts b. Find $div(\vec{F})$

$ze^y - xze^y + 1$

2pts c. Find $curl(\vec{F})$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xze^y & -xze^y & z \end{vmatrix} = xe^y\vec{i} + xe^y\vec{j} - (ze^y + xze^y)\vec{k}$$

2pts d. Is $\vec{F}$ a conservative vector field? Explain how you know.

No since $curl(\vec{F}) \neq 0$