More Examples of Nasty Ends

by F. C. Tinsley

(joint with C. R. Guilbault)

Introduction:

We construct an example of an end of a manifold that has semistable fundamental group and stable homology but is not inward tame. This example is constructed using the well known Baumslag-Solitar non-Hopfian group:

$$G = \left\langle a, b | a^3 = b^{-1} a^2 b \right\rangle$$

and its associated surjective homomorphism, $\phi : G \twoheadrightarrow G$, with non-trivial kernel. In fact, we build the example in high dimensional Euclidean space using the inverse limit of countably many iterations of this homomorphism.

Properties of the non-Hopfian Group:

1. $\phi(a) = a^2$ and $\phi(b) = b$ induce a homorphism of G onto G with non-trivial kernel.

2. The kernel of ϕ is generated as a normal subgroup by a single non-trivial element, $a^{-1}(a^{-1}b^{-1}ab)^2$. Equivalently, we may use the element, $\left[a, b^{-1}ab\right]$.

3. Let K be the 2-complex with two 1-cells and one 2-cell arising naturally from the presentation of G. There is a natural continuous map $f : K \to K$ that induces the homomorphism ϕ on fundamental groups. Now, f (or equivalently ϕ) induces isomorphisms on homology: $f_* : H_*(K; \mathbb{Z}) \xrightarrow{\approx} H_*(K, \mathbb{Z})$.

Construction of the End:

A: The Interation

Step 0: Embed K nicely in a high-dimensional Euclidean space, $\mathbb{R}^n$.

Step 1: Let $f : K \to K \subset \mathbb{R}^n$ be the map described above. Let f_1 be a nice embedding of K homotopic to f inside a small regular neighborhood N_0 in $\mathbb{R}^n$ of K. Let K_1 be the image of f_1 and N_1 a small regular neighborhood of K_1 in N_0.

Step m: Construct f_m, K_m, and N_m in the obvious way.

B. The Inverse Limit

The inverse limit, L, of the maps, $K \xleftarrow{f_n} K$ embeds naturally in $\{\bigcap N_m\} \subset \mathbb{R}^n$.

C. Properties of the Inverse Limit

1. For any number of reasons L is not an ANR. In fact, it is a generalization of the solenoid that we know and love in $\mathbb{R}^3$ arising from $\mathbb{Z} \xleftarrow{\times 2} \mathbb{Z}$. The argument in support of this fact is also a generalization of the one for $\mathbb{R}^3$.

2. The inclusion $j : L \to N_m$ induces isomorphisms on Čech homology.

Properties of the End

1. The end is not inward tame by Property C.1 above. An evident sequence of spheres that link L cannot be homotoped inside of $\mathbb{R}^n \backslash L$ into a compact subset.

2. The fundamental group of the end is "sort-of-stable" in that it is constructed of cobordisms whose boundaries have isomorphic fundamental groups and induce via inclusion isomorphisms on homology.

3. Whether there exists a surjection $\phi : G \to G$ with non-trivial perfect kernel and G finitely presented remains an extremely difficult unsolved problem.