F & F Problem: The Fundamental Group

In class on Friday we defined an equivalence relation on the set of paths $\alpha : [0,1] \to X$ where $(X, \mathcal{T})$ is a topological space and $\alpha(0) = x_0 \in X$ and $\alpha(1) = x_1 \in X$ are fixed. In particular, $\alpha \underset{x_0,x_1}{\sim} \beta$ if there exists a continuous function $F(s,t) : [0,1] \times [0,1] \to X$ (homotopy) such that $F(s,0) = \alpha(s)$, $F(s,1) = \beta(s)$, $F(0,t) = x_0$, and $F(1,t) = x_1$. We say that F deforms α to β fixing endpoints. We focus on the set of equivalence classes (homotopy classes) of paths joining x_0 to x_1.

If α and β are paths joining x_0 to x_1 and x_1 to x_2, then we defined a multiplication $\gamma = \alpha \cdot \beta$ where γ is a path joining x_0 and x_2. Moreover, we showed that this multiplication is well-defined with respect to equivalence classes. We also talked about inverses and an identity element. Your job is to put these together into a derivation of what is called the fundamental group of a topological space.

Definition 1. *Let $(X, \mathcal{T})$ be a topological space and $* \in X$. We define the "fundamental group of X based at $*$" which is denoted $\pi_1(X, *)$. The elements of the group are equivalence classes of paths $\alpha : [0,1] \to X$ where $\alpha(0) = \alpha(1) = *$. These paths often are called "loops" for obvious reasons. The multiplication of two elements $\alpha \cdot \beta$ is as given above except with $x_0 = x_1 = *$.*

Prove that $\pi_1(X, *)$ is a group under the operation $\alpha \cdot \beta$. Remember that the definition of a group requires proof that this operation is associative, that each element has an inverse, and that there is an identity element.