Math 375 Analysis Solutions to Practice Problems 2008-09 Tinsley

1. A subset E of a metric X space is compact if every sequence $\{x_n\}$ in E has a subsequence that converges in E. Suppose E is finite, i.e., $E = \bigcup_{k=1}^{n} \{a_k\}$ and let $\{x_n\}$ be a sequence of points in E. So, each x_n equals one of the points a_k of E. Let $E_k = \{n \in \mathbb{Z}^+ | x_n = a_k\}$. Then, $\bigcup_{k=1}^{n} E_k = \mathbb{Z}^+$. Since $\mathbb{Z}^+$ is infinite, one of the E_k's is infinite. Write this set E_k as $\{n_1, n_2, \cdots, n_j, \cdots\}$ where $n_j < n_{j+1}$ for all $j = 1, 2, \cdots, \infty$. The sequence $\{x_{n_1}, x_{n_2}, \cdots, x_{n_j}, \cdots\}$ converges since it is the constant sequence.

2. Let (X, d) be a metric space and $A = (\bigcup_{k=1}^{\infty} \{s_k\}) \bigcup \{s\}$ a subset of X. We must show that the complement of A (i.e., $X - A$) is open. Let $x* \in X - A$. In particular, $x* \neq s$. Let $\epsilon_1 = d(x*, s)/2$. There is an $n_0 \in \mathbb{Z}^+$ such that $d(s, x_n) < \epsilon_1$ for $n > n_0$. By the triangle inequality $x_n \notin B(x*; \epsilon)$. Let $\epsilon_2 = \min \{d(x*, x_1), d(x*, x_2), \cdots, d(x*, x_{n_0})\}$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. By construction $B(x*; \epsilon) \bigcap A = \emptyset$.

3. The proof assumes the definitions of limit inferior and limit superior.

   $b_n = \sup_{k \geq n}\{s_k\} \geq \inf_{k \geq n}\{s_k\} = c_n$
   Also, $b_n \geq b_{n+1}$ and $c_n \leq c_{n+1}$ for all n
   Suppose $\inf\{b_n\} < \sup\{c_n\}$; then there exists an m such that $b_m < \sup\{c_n\}$
   By a similar argument, there is a p such that $c_p > b_m$
   Let $q = \max\{p, m\}$; it follows that $c_q \geq c_p > b_m \geq b_q$ which contradicts $c_q \leq b_q$.