62

90

Math 126 Calculus I Final Exam Block 7 2008-09

81
72
58.5
49.5

55.8↑
49.6↑
40.3↑
34.1↑

Instructions: This part is a closed-book, closed-notes, closed calculator, closed computer, and closed-neighbor exam with the exception of two 3"x5" index cards on which anything may be written (both sides). Show all relevant work to receive full credit. Include a signed statement of the Honor Code. Part I of the exam is due one and three quarter hours after you begin. Turn it in to my office and pick up Part II.

1. (15 points) In each case, find $\frac{dy}{dx}$. You must have an expression for $\frac{dy}{dx}$, but you need not simply this expression.

65
90
5850
90
65
450
45

a. $y = x^3 - \int_{-2}^{2x} e^{-t^2}\,dt - \frac{3}{x^2} + \arcsin(3x) + C$

$y' = 3x^2 - e^{-(2x)^2}\cdot 2 + \frac{6}{x^3} + \frac{3}{\sqrt{1-9x^2}} + 0$

1 1.5 1 1 .5

55
62
110
330
65

65
62
130
390
40.3

b. $x^3 + \cos(xy) = \sqrt{y} - \ln(x) + \tan(y)$

.5 1 .5 1

$3x^2 - \sin(xy)(xy'+y) = \frac{1}{2}y^{-1/2}\cdot y' - \frac{1}{x} + \sec^2(y)\cdot y'$

$3x^2 - y\sin(xy) + \frac{1}{x} = \left(x\sin(xy) + \frac{1}{2}y^{-1/2} + \sec^2(y)\right)\cdot y'$

$y' = \frac{(3x^2 - y\sin(xy)) + 1/x}{x\sin(xy) + 1/2\,y^{-1/2} + \sec^2 y}$ 1

(1) prod
(1) quot

c. $y = \frac{(1+x^3)\sin(x)}{x+e^{-4x}}$

$y' = \frac{(x+e^{-4x})((1+x^3)\sin(x))' - (x+e^{-4x})'(1+x^3)(\sin(x))}{(x+e^{-4x})^2}$

$= \frac{[(x+e^{-4x})(3x^2\sin(x) + (1+x^3)\cos(x)) - (1-4e^{-4x})(1+x^3)(\sin(x))]}{(x+e^{-4x})^2}$

(1.5) 1

2. (25 points) Evaluate each of the following indefinite and definite integrals as completely as possible.

a. $\int\left(2x + 2x^{-1} + 4 - \frac{1}{1+x^2}\right)dx$

$= x^2 + 2\ln|x| + 4x - \tan^{-1}(x) + C$

b. $\int 2x^2\sqrt{x^3-8}\,dx = \int 2\frac{1}{3}\cdot du\, u^{1/2} = \int \frac{2}{3}u^{1/2}du$

$u = x^3 - 8$
$du = 3x^2dx$
$\frac{1}{3}du = x^2dx$

$= \frac{2}{3}\cdot\frac{u^{3/2}}{3/2} + C = \frac{4}{9}(x^3-8)^{3/2} + C$

25

c. $\int_1^4 \frac{3}{\sqrt{x}}dx = \int_1^4 3x^{-1/2}\,dx = \left.\frac{3x^{1/2}}{1/2}\right|_1^4$

$= \left.6x^{1/2}\right|_1^4 = 6\cdot 2 - 6\cdot 1 = 6$

d. $\int_0^{1/4} \frac{3}{\sqrt{1-4x^2}}dx = \int_0^{1/4} \frac{3\,dx}{\sqrt{1-(2x)^2}} = \int_0^{1/2} \frac{3}{2}\frac{du}{\sqrt{1-u^2}}$

$= \left.\frac{3}{2}\sin^{-1}(u)\right|_0^{1/2} = \frac{3}{2}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{4}$

$u = 2x$

$\frac{du}{2} = dx$

e. $\int \frac{e^{2x}}{4-e^{2x}}dx = \int \frac{du/(-2)}{u} = \frac{\ln|u|}{-2} + c$

$u = 4 - e^{2x}$

$du = -2e^{2x}dx$

$\frac{du}{-2} = e^{2x}dx$

$= \frac{\ln|4-e^{2x}|}{-2} + c$

3. (5 points) Let $f(x) = \sqrt{2x-1}$. Calculate $f'(x)$ directly from the limit definition of the derivative. No credit will be given for use of rules of differentiation. Show all steps. (Of course, you might want to check your answer using rules.)

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ (.5) $= \lim_{h\to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h}$ (1) $\frac{(\quad + \quad)}{(\quad + \quad)}$ (1)

$= \lim_{h\to 0} \frac{(2(x+h)-1 - (2x-1))}{h(\quad + \quad)}$ (1) $= \lim_{h\to 0} \frac{2h}{h(\sqrt{2(x+h)-1} + \sqrt{2x-1}}$ (1)

$= \frac{2}{2\sqrt{2x-1}} = \frac{1}{\sqrt{2x-1)}}$ (.5)

20

4. (5 points) An object moves along the number line with velocity $v = t + 2\cos(t)$.

a. Find the acceleration $a = \frac{dv}{dt}$ of this object as a function of time t.

$$a = v' = 1 - 2\sin(t)$$

(2)

b. Find the position $s = s(t)$ of this object along the number line as a function of time if $s = 2$ when $t = 0$.

$$s(t) = \int (t + 2\cos(t))\,dt = \frac{t^2}{2} + 2\sin(t) + C$$

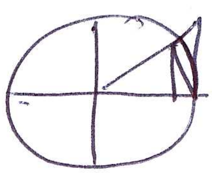
(2)

$$2 = s(0) = C \qquad s(t) = \frac{t^2}{2} + 2\sin(t) + 2$$

(1)

5. (5 points) Prove that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Do not use l'hopital's rule.

<u>$x > 0$</u>



$$A_1 < A_2 < A_3$$

$$\tfrac{1}{2}\cdot 1 \cdot \sin(x) < \tfrac{1}{2}x < \tfrac{1}{2}\tan(x) \qquad \text{Areas}$$

$$\frac{1}{\sin(x)} > \frac{1}{x} > \frac{\cos(x)}{\sin(x)} \qquad \text{Alg}$$

$$1 > \frac{\sin(x)}{x} > \cos(x) \qquad \text{alg}$$

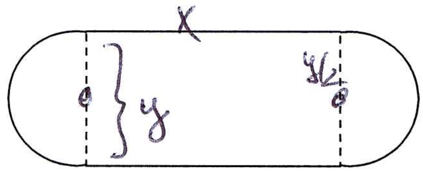
Take limit

$$1 \geq \lim_{x\to 0} \frac{\sin(x)}{x} \geq \lim_{x\to 0} \cos(x)$$

$$1 = \lim_{x\to 0} \frac{\sin(x)}{x} \qquad \text{(squeeze)}$$

16

6. (7 points) A running track is to be built in the shape of a rectangle surmounted on opposite sides by a pair of semi-circles. The length of the track (the perimeter only) is to be 500m.



Find the dimensions of the rectangle that maximize the **area of the rectangle** (not the total area inside the track). Be sure to justify that that you, indeed, have found a maximum. The area of a circle is $\pi \cdot r^2$, the area of a rectangle is base times height, and the perimeter of a circle is $2\pi r$.

$A = xy$ (.5)

$500 = 2x + \cancel{2}\pi(y/\cancel{2})$ (1)

$y = \dfrac{500-2x}{\pi}$

$A = x\left(\dfrac{500-2x}{\pi}\right)$ (1) $\quad 0 \le x \le 250$ (1)

A=0 A=0

$= \frac{2}{\pi}x(250-x)$

② $A' = \frac{2}{\pi}(250-2x) = -\frac{4}{\pi}(x-125)$ (1)

$-\frac{4}{\pi}$ – – – – – – – – –

$x-125$ – – – 0 + + + +

A' + + + + 125 – – – –

0

inc dec

global max (1.5)

① = 0 $x = 125$ $y = \frac{250}{\pi}$ (1)

A Must be a max.

③ $A'' = -\frac{4}{\pi}$ Concave down ⇒ global max

7

28

Math 126 Calculus I Final Exam Block VII 2008-09 Part II

25.2
22.4
18.2
15.4

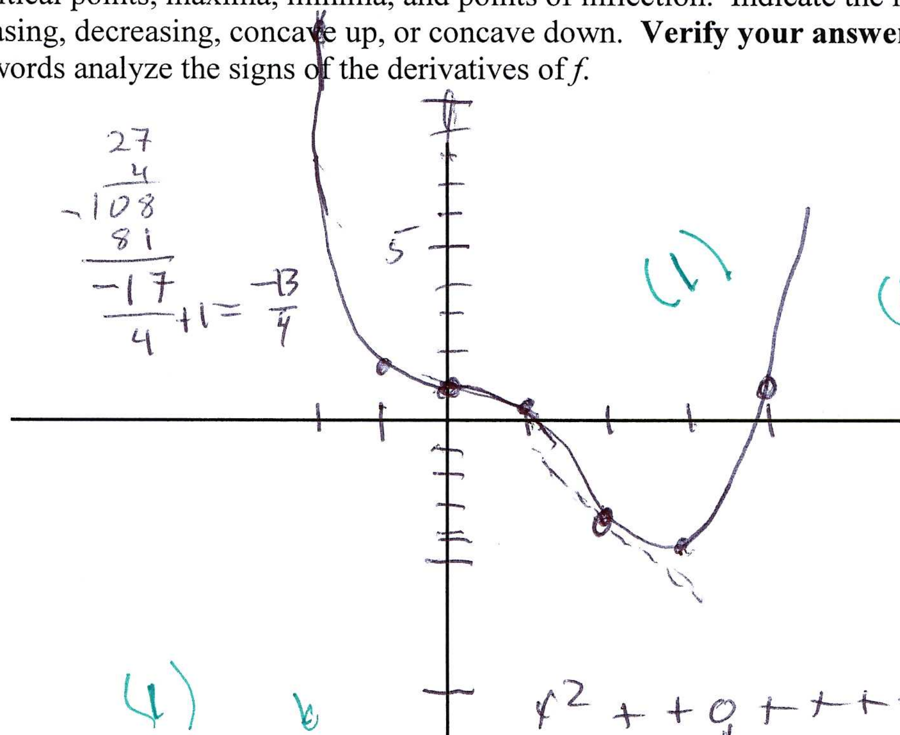
Instructions: This is a closed-book, closed-notes, and closed-neighbor exam with the exception of your two 3”×5” index cards. You may use your calculator or Derive (the Tutt Science 213 lab should be open). Show all relevant work to receive full credit. Include a signed statement of the Honor Code. This part of the exam is due one and one quarter hours after you begin.

7. (8 points) You may use DERIVE or a calculator to check your work on this problem.

Consider the function $y = f(x) = \frac{x^4}{4} - x^3 + 1$. Carefully sketch the graph. Label all stationary points, critical points, maxima, minima, and points of inflection. Indicate the intervals on which f is increasing, decreasing, concave up, or concave down. **Verify your answers using calculus**. In other words analyze the signs of the derivatives of *f*.

x	y
-1	7/4
0	1
1	1/4
2	-3
3	-13/4
4	1
-2	3

27
4
-108
81
$\frac{-17}{4} + 1 = \frac{-13}{4}$



.5 min

(1) crit x = 0, 3

(1) Inf x = 0, 2

(1)

$y' = x^3 - 3x^2 = x^2(x-3)$

$y'' = 3x^2 - 6x = 3x(x-2)$

x^2 + + 0 + + + + + + + + + 1.5

x-3 − − | − − 0 + + + +

y' dec | dec 3 inc

0 crit 3 loc min

3x − − | + + + + + +

x-2 − | − − 0 + + + + 1.5

A'' + + + 0 cd 2 + + + cu

cu Inf Inf

7 28 × 9 = 25.2

6 28 × 8 = 22.4

28 × 65 = 140 + 168 = 1820

28 × 55 = 140 + 1400 = 1540

6

8. (5 points) In this problem, you again will be working with the function in problem 7. It is given by $y = f(x) = \frac{x^4}{4} - x^3 + 1$. The y intercepts or roots of this function cannot be found easily because formulas for the roots of a quartic equation are extremely complicated (that's why I did not ask you to find all intercepts!).

a. Use Newton's method to approximate one of the roots. Start with the point $x_0 = 2$ and complete two iterations (ie, you will find x_2). Locate your approximate root on your graph in problem 7. $X_0 = 2$

$5/4 \quad \frac{3149}{2800}$

$y' = x^3 - 3x^2$

$2, 1.25, 1.1246$

$y'|_{x=2} = -4$

$X_1 = 2 - \frac{-3}{-4}$

$y|_{x=2} = -3$

$y'|_{x=1.25} = \cancel{2.734}\ -2.734$

$= 1.25$

$y|_{x=1.25} = -.3428$

$X_2 = 1.25 - \frac{-.3428}{-2.734} =$

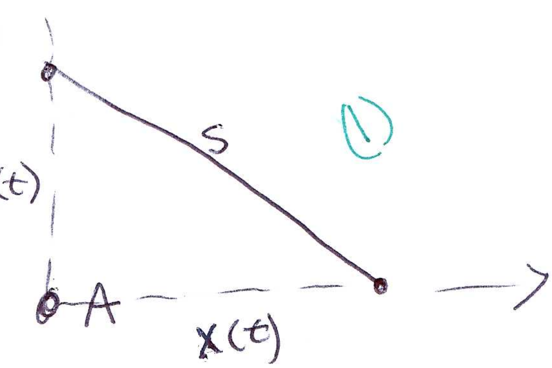
b. Consider your graph for problem 7. Does $y = f(x)$ have more than one root? If so, what different starting points might you choose to find these roots? Why? (You do not have to actually approximate these roots.)

yes; $f(X_*) = 0$ for $3 < X_* < 4$. I would try $X_0 = 4$. X_0 is close to the root & the tangent line will give the next X_1 even closer

9. (6 points) A car leaves city A at noon traveling north at a rate of 50 mph. A bus leaves city A at 1:00 PM on the same day traveling east at 60 mph. At what rate is the distance between the car and the bus increasing at 2:30 PM? Be sure to show and label all work to receive maximal credit.

$y(t) = \text{dist}(A, \text{Car})(t)$

$x(t) = \text{dist}(A, \text{bus})(t)$



Instead $t = 0$ at noon

.5

$t = 2.5$

(1)

$y = 50t$ (1) $\quad t \geq 0$

$X = 90\text{mi}$

$y = 125\text{mi}$

$x = \begin{cases} 0 & 0 \leq t \leq 1 \\ 60(t-1) & t \geq 1 \end{cases}$

$s^2 = x^2 + y^2$ (1)

$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt} + \frac{y}{s}\frac{dy}{dt}$ (1.5)

$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + (1)\ 2y\frac{dy}{dt}$

$= \frac{90 \cdot 60}{\sqrt{90^2+125^2}} + \frac{125 \cdot 50}{\sqrt{90^2+125^2}} = 75.64$

6

10. (6 points) The standard "bell" curve is the graph of the function $\mathbf{f(x)} = \frac{1}{\sqrt{2\pi}}\mathbf{e}^{-\frac{x^2}{2}}$. The probability that *a standard normal variable* falls within a specific interval is measured by the area between this curve and the *x*-axis over that interval. Approximate the probability that such a variable falls between 0 and 1. Use 5 rectangles with heights taken at the right-hand endpoints (i.e., R_5). Make a relevant sketch (feel free to copy the sketch of **f(x)** from your calculator or from Derive). Does your approximation under or over estimate the area? Explain.

.5

(5) (2) (1)

$R_5 = (.2)(f(.2)+f(.4)+\dots+f(1))$

(1)

$= (.2)(.3910 + .3683 + .3332 + .2897, 0.2420)$

$= .3248$ (1)

$= (.2)(1.6242)$

(1) underestimate f is dec.

11. 3 free points. Enjoy your break!

9