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Math 126 Calculus I Midterm Exam Block 3 2009-10 Part I

424

Write down the time that you begin the exam: ____________________

Instructions: This is a closed-book, closed-notes, closed calculator, closed computer, and closed-neighbor exam with the exception of one 3"x5" index card on which anything may be written. Show all relevant work to receive full credit. Include a signed statement of the Honor Code. Part I of the exam is due one hour and forty five minutes after you begin after which you can pick up Part II.

90 ~~64~~ ↑ 5
80 ~~57~~ ↑ 11
65 ~~46~~ ↑ 5
55 ~~39~~ ↑ 3
3

1. (5 points)

a. State the limit definition of the **derivative** of a function $y = f(x)$.

.5 $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

b. Let $y = f(x) = \frac{1}{\sqrt{3x+1}}$; calculate the derivative of $f(x)$ **directly from the limit definition**. You will receive no credit for calculating the derivative using rules.

4.5

(.5) (1) (1)

$$\frac{\frac{1}{\sqrt{3(x+h)+1}} - \frac{1}{\sqrt{3x+1}}}{h} = \frac{\sqrt{3x+1}-\sqrt{3(x+h)+1}}{h\sqrt{3(x+h)+1}\sqrt{3x+1}} \cdot \frac{(\sqrt{3x+1}+\sqrt{3(x+h)+1})}{(\text{same})}$$

$$= \frac{3x+1-(3(x+h)+1)}{h\sqrt{3(x+h)+1}\sqrt{3x+1}\,(\text{same})} = \frac{-3h}{h\sqrt{3(x+h)+1}\sqrt{3x+1}(\sqrt{3x+1}+\sqrt{3(x+h)+1})}$$

$$\searrow \frac{-3}{2(3x+1)^{3/2}} = -\frac{3}{2}(3x+1)^{-3/2}$$

2. (20 points) In each case, find $\frac{dy}{dx}$ using any method. You must find an expression for $\frac{dy}{dx}$, but **you need not simplify your answers**. There are two more parts on the next page.

a. $y = x^3 - \frac{4}{\sqrt{x}} + \frac{2}{x} - ln(x) + 4 - sin^{-1}(3x)$

$$3x^2 + 2x^{-3/2} - \frac{2}{x^2} - \frac{1}{x} - \frac{1}{\sqrt{1-(3x)^2}} \cdot 3$$

b. $y = e^{(e^x)}$

$$y' = e^{(e^x)} \cdot e^x = e^{e^x + x}$$

15

c. $y = tan^{-1}(x^2)\ln(x^4+1)$

$\left(\frac{2x}{1+x^4}\right)\ln(x^4+1) + \left(\tan^{-1}(x^2)\,\frac{4x^3}{x^4+1}\right)$ Same 2

1.5 1.5

d. $y = (\tan(x))^{\cos(3x)}$

$\ln y = \cos(3x)\ln(\tan(x))$ Form 2

$\frac{1}{y}y' = -3\sin(3x)\ln(\tan(x)) + \cos(3x)\cot(x)\cdot\sec^2(x)$

$y' = (\tan(x))^{\cos(3x)}\left(-3\sin(3x)\ln(\tan(x)) + \cos(3x)\cot(x)\sec^2(x)\right)$ (1)

4. (5 points) Prove the product rule for derivatives. Indicate a reason for each step.

$f'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

trick $= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$

sum thm for limits $\lim_{h\to 0} \frac{(f(x+h)-f(x))g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h)-g(x))}{h}$

Prod & quot thm's for lim $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x)\cdot \lim_{h\to 0} \frac{(g(x+h)-g(x))}{h}$

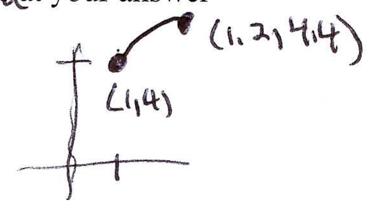
defn $= f'(x)\,g(x) + f(x)\cdot g'(x)$

5. (3 points) Suppose $f(1) = 4$, $f'(1) = 2$, and $f''(1) = -1$.

a. Find an approximation for $f(1.2)$. Explain how you arrived at your answer.

$\Delta f \doteq f'(1)\cdot\Delta x = 2\cdot(.2) = .4$ $f(1.2) \doteq 4 + .4 = 4.4$

b. Sketch the section of the graph of f that joins $(1,4)$ to $(1.2, f(1.2))$. Again, explain how you arrived at your answer



$f'(1)$

Concave down.

18

6. (5 points) On the axes below, sketch the graph of a function, $y = f(x)$, (yes, there are many possibilities) that satisfies all of the following conditions simultaneously. Label your graph appropriately. You graph must **clearly** exhibit each of following properties.

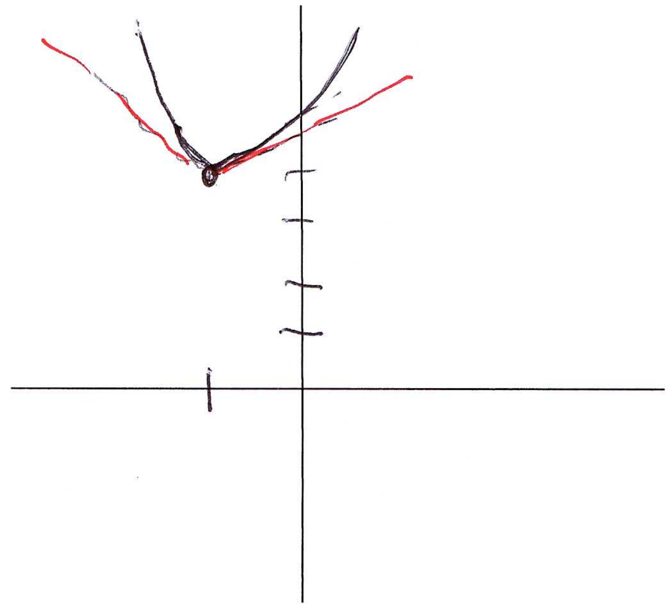
$f(-1)=4$ is a local minimum

$f''(x)>0$ for $x \neq -1$

$f'(x)<0$ for $x<-1$

$f'(x)>0$ ~~for $x<-1$~~ $x>-1$

$f'(x)$ is undefined for $x=-1$



7. (6 points) Use implicit differentiation to find $\frac{dy}{dx}$. You must solve for $\frac{dy}{dx}$, but you need not simplify your answer: $\ln(y^2+1)+x^3=\tan(x^2y^2)+e^2-y$

$$\frac{1}{y^2+1}\cdot 2y\cdot y' + 3x^2 = \sec^2(x^2y^2)\,(2xy^2+2x^2y\cdot y') - y'$$

1.5 .5 1 1.5 .5

$$3x^2-2xy^2\sec^2(x^2y^2) = \left(2x^2y\sec^2(x^2y^2)-\frac{2y}{y^2+1}-1\right)y'$$

$$y' = \frac{3x^2-2xy^2\sec^2(x^2y^2)}{2x^2y\sec^2(x^2y^2)-\frac{2y}{y^2+1}-1}$$ (1)

39.5↑ A

35.0↑ B

28.5↑ C

22 ↑ D

21.5↓ NC

11

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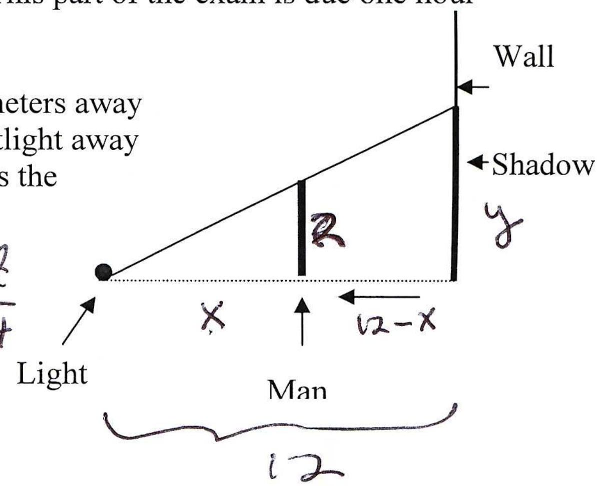
Math 126 Calculus I Midterm Exam Block III 2009-10 Part II

Record what time you begin this part: ________________

Instructions: This is a closed-book, closed-notes, and closed-neighbor exam with the exception of your 3"×5" index card. You also may use your calculator. Show all relevant work to receive full credit. Include a signed statement of the Honor Code. This part of the exam is due one hour and thirty minutes after you begin.

8. (8 points) A spotlight on the ground shines on a wall 12 meters away from the light. If a man, 2 meters tall, walks toward the spotlight away from the wall at a speed of 1.6 meters per second, how fast is the length of his shadow on the wall increasing when he is 4 meters from the building?



Vars: t - sec; $x = d(\text{man}, \text{light})$; y = length of sh.

Constants (2): $d(\text{light}, \text{wall}) = 12\,m$; ht $= 2\,m$

Ts last: $12 - x = 4$, $x = 8$

$\frac{x}{2} = \frac{12}{y}$ (.5)

$xy = 24$ (.5)

$x\frac{dy}{dt} + \frac{dx}{dt}\cdot y = 0$ (1.5)

when $x = 8, y = 3$ (.5)

$x\cdot\frac{dy}{dt} + (-1.6)\cdot y = 0$ (1)

$\frac{dy}{dt} = \frac{1.6\,y}{x} = \frac{1.6\cdot 3}{8} = \frac{4.8}{8} = .6$ m/sec (1)

9. (8 points) Physicists and a few economists commonly use a function called the *ERF* function. It usually is given by its derivative and a single initial condition:

$$\frac{d}{dx}(ERF(x)) = \frac{2}{\sqrt{\pi}}e^{-x^2} \text{ and } ERF(0) = 0$$

The domain of the ERF function is the set of all real numbers. Don't be confused by the notation ERF. This is just a name, like ***ln(x)***, ***sin(x)***, or ***cos(x)***.

a. Calculate $\frac{d^2}{dx^2}(ERF(x))$, the second derivative of $ERF(x)$ with respect to x.

$= \frac{2}{\sqrt{\pi}}(-2x)e^{-x^2}$

cu Po Inf cd

$++++\ 0\ ----$

0

b. Find the intervals over which $ERF(x)$ is increasing or decreasing. Give reasons for your answers.

$\frac{2}{\sqrt{\pi}}e^{-x^2} > 0 \Rightarrow$ always inc.

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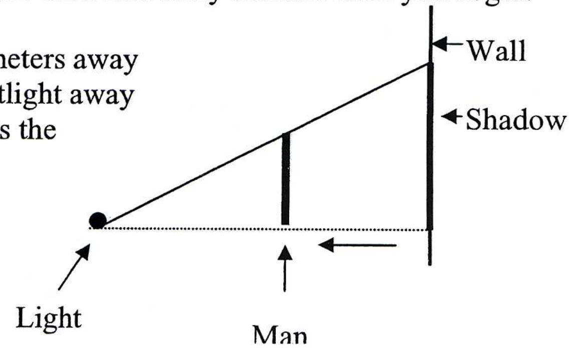


9. (8 points) Physicists and a few economists commonly use a function called the *ERF* function. It usually is given by its derivative and a single function value (also called an initial condition):

$$\frac{d}{dx}(ERF(x)) = \frac{2}{\sqrt{\pi}}e^{-x^2} \text{ and } ERF(0) = 0$$

The domain of the *ERF* function is the set of all real numbers. Don't be confused by the notation *ERF*. This is just a name, like ***ln(x)***, ***sin(x)***, or ***cos(x)***.

a. Find the intervals over which $ERF(x)$ is increasing or decreasing. Give reasons for your answers.

(2) $(-\infty, \infty)$ increasing $\frac{2}{\sqrt{\pi}}e^{-x^2} = f'(x) > 0$

b. Are there any critical points? If so, label them as local maxima, local minima, or points of inflection. Give reasons for your answers.

(1) No; $f'(x) > 0$ always

c. Calculate $\frac{d^2}{dx^2}(ERF(x))$, the second derivative of $ERF(x)$ with respect to x.

$$\frac{d}{dx}(f'(x)) = \frac{d}{dx}\frac{2}{\sqrt{\pi}}e^{-x^2} = \frac{-4xe^{-x^2}}{\sqrt{\pi}} \quad (1)$$

d. Find the intervals over which $ERF(x)$ is concave up or concave down. Locate any points of inflection not found in (c). Again, give reasons for your answers.

$-4/\sqrt{\pi}$ — — — — — — —

x − − − 0 + + + +

e^{-x^2} + + + + + + +

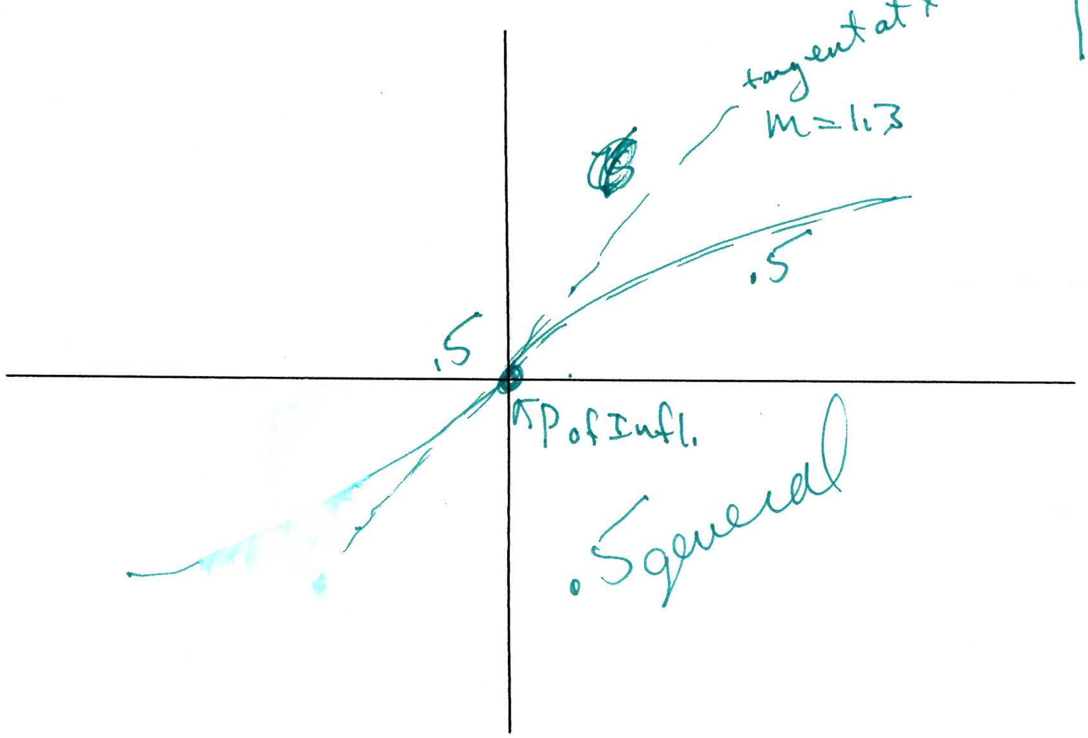
+ + + + 0 − − − −

cu 0 cd 0 (1.5)

0 is a P of Infl. (1)

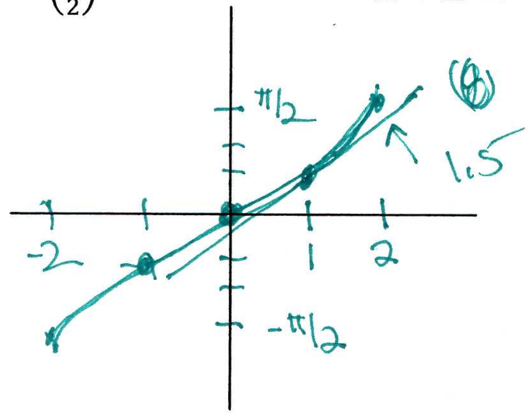
e. Sketch an approximate graph for $ERF(x)$. Use what you know about derivatives and second derivatives and that $ERF(0) = 0$.

x	$f(x)$	$f'(x)$
0	0	$2/\sqrt{\pi} \approx 1.13$



10. (4 points)

a. Sketch the curve $\sin^{-1}\left(\frac{x}{2}\right)$ over the interval $-2 \le x \le 2$.



b. Find the equation of the tangent line to the curve $y = \sin^{-1}\left(\frac{x}{2}\right)$ at $x = 1$. Leave it in point-slope form.

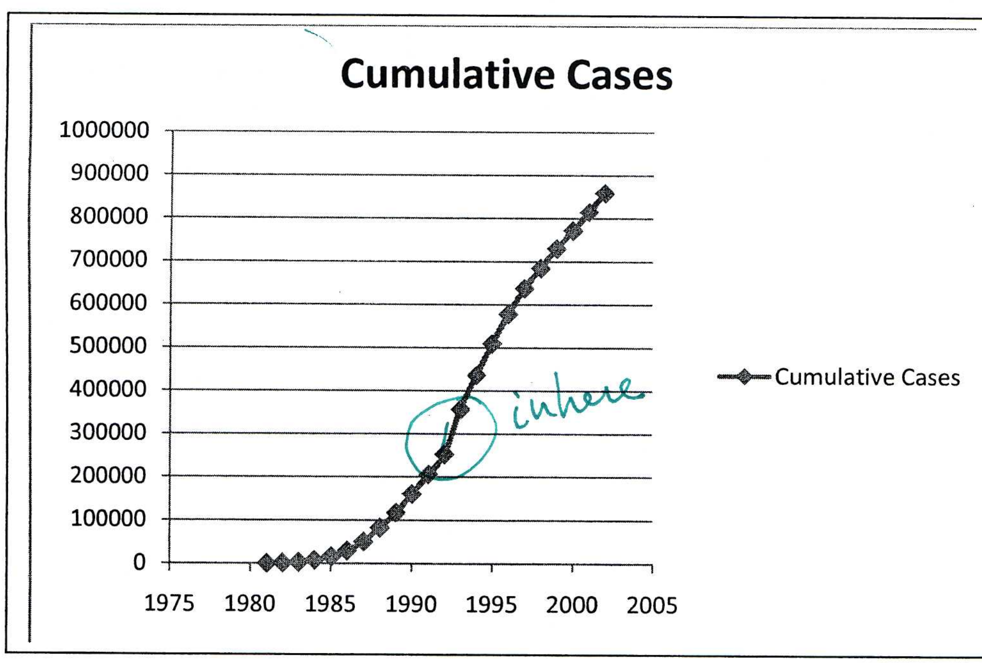
$\left(y - \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}(x-1)$ (1) $\quad \sin^{-1}\left(\frac{1}{2}\right) = \pi/6 = y$ (.5) $\quad y' = \frac{1}{2}\frac{1}{\sqrt{1-\frac{x^2}{4}}}\Big|_{x=1}$ (1) $= \frac{1}{2\sqrt{3/4}} = \frac{1}{\sqrt{3}}$

.524 .577

c. Place your line from part (b) in your sketch from part (a). Label your sketch appropriately.

11. (7 points) In class we have been studying the early years of the AIDS epidemic in the United States. We have observed that the incidence of AIDS was increasing at an increasing rate. The Centers for Disease Control now provides the following data through 2002. Lets agree to call this function *f*.

Year	Cumulative Cases
1981	155
1982	765
1983	2782
1984	7192
1985	15447
1986	28619
1987	50271
1988	82143
1989	117076
1990	160199
1991	205361
1992	252645
1993	357358
1994	436338
1995	509354
1996	577486
1997	637248
1998	684765
1999	730639
2000	772311
2001	815047
2002	858997



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11. (Continued)

a. Based on the plot on the previous page, what year looks like it might correspond to a point of inflection? Explain your answer and label this point on the graph.

1991– 1993

b. Verify your answer by computing relevant numerical derivatives. Show your work below. Explain your answer

	f	f'	f''
89	117076		
90	160199	44142.5	
91	205361	46223	15928
92	252645	75998.5	22811.75
93	357358	91846.5	~~[illegible]~~ -.025
94	436338	75998	-~~[illegible]~~ -10636.25
95	509354	70574	
96	577486		

24.5↑

21.5↑

17.5↑

15.0↑