Necessary Pro-π_1 Conditions for Inward Tame Ends

F. C. Tinsley (with Craig Guilbault)

July 10, 2008

0 Introduction

We continue our study of the ends of high-dimensional manifolds. For the sake of simplicity, we restrict our attention to **one-ended** manifolds of dimension at least six. If M is such a manifold, then it contains arbitrarily large compacta with connected complement. M is *inward tame* if the complements of these compacta can be homotoped into a compact subset. Earlier, we identified two algebraic necessary conditions for a one-ended manifold to be inward tame: the pro-fundamental group sequence of the end must be semi-stable, and the pro-homology sequence of the end must be stable. In the proceedings of this workshop in 2005 (see [3]) we constructed a one-ended manifold that satisfies both of these algebraic conditions but is not inward tame. The purpose of this contribution is to outline a proof that our construction produces a manifold that is not inward tame where the proof depends only on pro-π_1 considerations.

1 Necessary Conditions on Pro-π_1

Earlier, we have generalized the results of Siebenmann to show that an inward tame end gives rise to an inverse sequence of finitely presented groups and surjective homomorphisms:

$$A_1 \overset{\lambda_2}{\twoheadleftarrow} A_2 \overset{\lambda_3}{\twoheadleftarrow} A_3 \overset{\lambda_4}{\twoheadleftarrow} \cdots$$

Each group corresponds to the fundamental group of a nice neighborhood of the end. In [1] we showed that the homology of the end is stable. Duality arguments in the universal cover of the end give some information about the fundamental groups and first homology as well. It is possible to choose nice neighborhoods so that $ker(\lambda_j) \subset [ker(\lambda_{j-1} \circ \lambda_j), ker(\lambda_{j-1} \circ \lambda_j)]$ for all $j \geq 2$.

We give an example of an end in the next section that cannot possibly satisfy this condition on fundamental groups. Yet, it exhibits both semi-stable fundamental groups and stable homology groups.

2 Non-Hopfian Baumslag-Solitar Groups

Our earlier construction derived from the Baumslag-Solitar group $G = \langle a, b | a = a^{-2}b^{-1}a^2b \rangle$. The associated surjective homomorphism $\phi : G \twoheadrightarrow G$ is given by $\phi(a) = a^2, \phi(b) = b$ and has non-trivial kernel generated normally by the single word $a^{-1}[a, b]^2 \neq 1 \in G$. In fact, it uses the inverse sequence:

$$G \overset{\phi}{\twoheadleftarrow} G \overset{\phi}{\twoheadleftarrow} G \overset{\phi}{\twoheadleftarrow} G \cdots$$

as a model. These homomorphisms are induced naturally on fundamental groups of the obvious two complexes by cellular maps. In turn, we use regular neighborhood theory to realize a one-ended manifold with the right algebraic properties in a high-dimensional Euclidean space.

The following lemma gives an indication as to why this end cannot be inward tame.

Lemma 2.1. *In the above setting* $ker(\phi) \nsubseteq [ker(\phi^2), ker(\phi^2)]$

Proof. We outline the steps in the proof.

1. The group G contains no non-trivial perfect subgroups. We use subgroup theorems for HNN extensions and an argument similar to but easier to that in [1].

2. $ker(\phi)$ is normally generated by the element $k_1 = a^{-1}[a, b]^2$

3. $ker(\phi^2)$ is normally generated by the two elements $k_1, k_2 = [b, a]^2(a^{-1})^b a^{b^2}(a^{-1})^b a^{b^2}$; note $\phi(k_2) = k_1$

4. If $ker(\phi) \subseteq [ker(\phi^2), ker(\phi^2)]$ then $\bigcup_n \{ker(\phi^n)\}$ is a non-trivial perfect subgroup of G. This contradicts (1). The proof is elementary and relies on careful use of Tietze transformations of presentations of G.

□

3 Closing

This particular example of an end cannot be inward tame. However, the **Z**-homology of the end is stable and the fundamental group of the end is semi-stable as are always the cases for inward tame ends. Because of the non-Hopfian property of G, the fundamental group of the end is in some sense constant. An example of the following type would be of great interest to us (and to many others!):

Question 3.1. *Does there exist a non-Hopfian finitely present group H with associated surjective homomorphism $\phi : H \to H$ with $ker(\phi)$ a non-trivial perfect normal subgroup of H?*

References

[1] C. R. Guilbault and F. C. Tinsley. Manifolds with non-stable fundamental groups at infinity, ii. *Geometry & Topology*, 7:255–286, 2003.

[2] C. R. Guilbault and F. C. Tinsley. Manifolds with non-stable fundamental groups at infinity, iii. *Geometry & Topology*, 10:541–556, 2006.

[3] F. C. T. (with C. R. Guilbault. More examples of nasty ends. *Proceedings of 22nd Annual Western Topology Workshop*, 2005.