Neighborhoods of crumpled manifold boundaries

by F. C. Tinsley

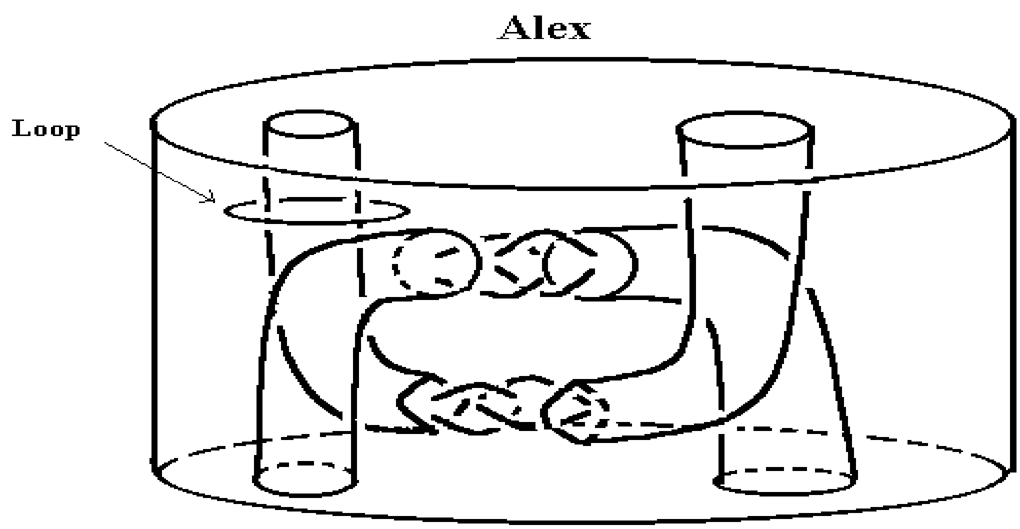
I. Setting: Let S^n denote the n-sphere and let $h : S^{n-1} \to S^n$ be an embedding. Identify S^{n-1} with $h(S^{n-1})$. In other words, assume S^{n-1} sits in S^n. Then, S^{n-1} separates S^n into two components. Denote the closure of one of these components by C^n. We call C^n a *crumpled cube*. Observe that $bdy(C^n) = S^{n-1}$ naturally.

EXAMPLE 1: Think of S^{n-1} in $\mathbb{R}^n$ as the set of points a distance one from the origin and S^n as the one-point compactification of $\mathbb{R}^n$. Then the crumpled cube containing the origin is, in fact, a cell.

DEFINITION 1: S^{n-1} is *tame in* C^n if C^n is a cell. Otherwise, S^{n-1} is *wild in* C^n.

We also may refer to a cell as a *trivial* crumpled cube.

EXAMPLE 2: The most famous example of a wild sphere is Alexander's Horned Sphere ($n = 3$) shown below in $\mathbf{R}^3$. S^2 is wild in its bounded component, C^3. Specifically, its interior, $C^3 \backslash S^2$, is not simply connected and so cannot be the interior of a cell. The loop labeled by an arrow does not bound a disk in C^3 missing S^2.



II. Some History: The study of wild embeddings of spheres has a rich history extending back more than half a century. In particular, myriads of necessary and sufficient conditions for a crumpled cube to be a cell have been developed. Morton Brown developed the following characterization that is valid in all dimensions:

PROPOSITION 1: A crumpled cube, C^n, is a cell if and only if S^{n-1} is *collared* in C^n, ie, if and only if there is an embedding $c : S^{n-1} \times [0,1] \to C^n$ with $c_0|S^{n-1}$ the identity.

A second, highly useful characterization was developed. This result is due to R. H. Bing in dimension 3, Frank Quinn in dimensions 4 and 5, and Bob Daverman in dimensions 6 and higher.

PROPOSITION 2: A crumpled cube, C^n, is a cell if and only if S^{n-1} has *1-LCC complement* in C^n, ie, given any $x \in S^{n-1}$ and $\epsilon > 0$ there is a $\delta > 0$ so that loops in $N_\delta(x, C^n) \backslash S^{n-1}$ bound 2-disks in $N_\epsilon(x, C^n) \backslash S^{n-1}$.

In short, C^n is a cell if and only if small loops near S^{n-1} and missing S^{n-1} bound small disks near S^{n-1} and missing S^{n-1}.

One obviously necessary condition for tameness still remains a candidate for also being a sufficient condition. S^{n-1} is *free* in C^n if for each $\epsilon > 0$ there is a map $f_\epsilon : S^{n-1} \to C^n \backslash S^{n-1}$ with $d(x, f(x)) < \epsilon$ for all $x \in S^{n-1}$.

FREE SPHERE QUESTION: Suppose S^{n-1} is free in C^n. Is C^n a cell?

III. Strategy of Investigation:

For $n = 3$, the free surface question is an extremely difficult and well-known unsolved problem. Also, Bob Daverman has developed methods for "inflating" examples of crumpled cubes from dimension n to dimension $n+1$. These facts suggest that the answer to the question is YES for $n = 3$ and NO for $n > 3$. From 1985-1995, Daverman and I constructed many new, intrinsically high-dimensional examples of non-trivial crumpled cubes. This research focuses on whether our new knowledge has anything to say about the free surface question in high dimensions.

We begin with what may be an easier question. For $n = 3$ it is well known that if S^2 is free in C^3, then $C^3 \backslash S^2$ is homeomorphic to an open 3-cell. The proof relies on the Sphere Theorem, an intrinsically 3-dimensional result.

POSSIBLY EASIER QUESTION: Suppose $n > 3$ and S^{n-1} is free in C^n, then is $C^n \backslash S^{n-1}$ homeomorphic to an open (n-1)-cell?

IV. Some Progress:

We focus for a moment on a small loop, s, in $C^n \backslash S^{n-1}$ near S^{n-1} and categorize what s may bound in order of increasing nastiness.

1. s bounds a *small disk* in $C^n \backslash S^{n-1}$ near S^{n-1}.
2. s bounds a *large disk* in $C^n \backslash S^{n-1}$ near S^{n-1}.
3. s bounds a *half-open annulus* properly embedded in $C^n \backslash S^{n-1}$ near S^{n-1}.
4. s bounds a *small disk with a Cantor set's worth of holes* properly embedded in $C^n \backslash S^{n-1}$ near S^{n-1}.

Comments:

1. This is the 1-LCC complement condition referred to above. So, S^{n-1} is tame in C^n.

2. This condition implies that $C^n \backslash S^{n-1}$ is homeomorphic to the interior of an n-disk. It often is described by saying $C^n \backslash S^{n-1}$ *is 1-LC at infinity.*

3. Alternatively, we may say that s can be "tubed to infinity". In our setting, this is equivalent to $E^n \backslash S^{n-1}$ being *outward tame*, ie, closed subsets of $E^n \backslash S^{n-1}$ near S^{n-1} can be homotoped in $E^n \backslash S^{n-1}$ arbirarily close to S^{n-1}. (See the article by Craig Guilbault in these proceedings.)

4. In general, this is the most we can hope for. However, a bit more is true. Any loop, s, can be made to bound a 2-disk in C^n so that the preimage in this disk of its intersection with S^{n-1} is a compact 0-dimensional set.

We show that crumpled cubes that belong to our category 3 and have S^{n-1} free in C^n also have Euclidean interior.

Proposition 3: Suppose S^{n-1} is free in C^n and given any neighborhood U of S^{n-1} in C^n there is a neighborhood V of S^{n-1} in C^n such that any loop s in V can be tubed to infinity in U. Then $C^n \backslash S^{n-1} \cong \mathbb{R}^n$.

Proof: We need to show that loops close to S^{n-1} in $C^n \backslash S^{n-1}$ bound disks close to S^{n-1} in $C^n \backslash S^{n-1}$ (the 1-LC at ∞ condition).

To this end, let U be an arbitrary neighborhood of S^{n-1} in C^n. Let V be the neighborhood

S^{n-1} in C^n satisfying the hypthesis of this theorem. Let s be an arbitrary loop in $V \backslash S^{n-1}$. Then, s bounds a half-open annulus, $A \cong S^1 \times [0, \infty)$, entirely contained in U and properly embedded in $C^n \backslash S^{n-1}$.

Let $\epsilon > 0$ be a small positive number and, by freeness, let $f : S^{n-1} \to C^n \backslash S^{n-1}$ be an ϵ-map. If ϵ is small enough, then $f(S^{n-1}) \subset V$, $f(S^{n-1})$ will separate s from S^{n-1}, and f will be a degree one map. Assume that f is in general position with respect to itself and A. Since A is embedded, $f^{-1}(A)$ is a finite union of simple closed curves in S^{n-1}, say $s_1, s_2, \cdots, s_m$. Finally, for at least one j, $f|s_j : S^1 \to A$ is essential at the π_1 level since f is of degree one (we the details of this argument to the reader).

Let $p : \tilde{U} \to U$ be the universal covering space of U. Since S^{n-1} is simply connected, the map f lifts to a map $\tilde{f} : S^{n-1} \to \tilde{U}$. Consider a component, $\tilde{A}$, of $p^{-1}(A)$. Now, $\tilde{A}$ must be either a half-plane or again an annulus. But, $\tilde{A}$ cannot be a half-plane because, on the one hand, $f|s_j$ is essential and would lift to a line but, on the other hand, $f|s_j$ must lift to a loop since $s_j \subset S^{n-1}$ and f lifts. Thus, $\tilde{A}$ is a half-open annulus.

Then $f|\tilde{A} : \tilde{A} \to A$ is a k to 1 map for some positive integer k. We argue that $k = 1$. Let $\alpha = * \times [0, \infty)$ be an arc in A running from $* \in s$ out to S^{n-1}. Without loss of generality, assume f and α are in general position so that their intersection is a finite number of points. Using orientations on S^{n-1}, C^n, and α, we may assign a $+1$ or a -1 to each point of intersection. Since f is of degree one, the sum of these must be $+1$ or -1. However, since $f|s_i$ lifts to $\tilde{A}$ for each i, $1 \le i \le m$, the sum must be congruent to 0 mod k. The only possibility is for $k = 1$.

V. Closing:

What makes our category 3 tractible is that the annulus, A, is a has an abelian fundamental group ($\mathbb{Z}$). As a result, study of the covering spaces of A is straightforward. The situation in general is considerably more complicated. We are led to understanding the intersections between f and disks with more than one hole. These objects have free fundamental groups with more than one generator and, thus, have a plethora of covering spaces.

The possible advantage to this complexity is that it may make aid in finding a counterexample. We close with a specific intersection pattern that would allow us possibly to use the our constructions referred to above. We abbreviate the commutator of two group elements, a and b, by $[a, b]$, ie, $a^{-1}b^{-1}ab = [a, b]$.

Final Question: Let G be Higman's group presented with four generators and four relators as

$$\langle a_i | a_i = [a_i, a_{i+1}], a_5 = a_1 \;\; 1 \le i \le 4 \rangle$$

Is there a non-trivial crumpled cube C^n with S^{n-1} free in C^n, a loop s bounding a disk with four holes H, and a map $f : S^{n-1} \to C^n \backslash S^{n-1}$ whose intersection pattern with H yields Higman's group?

We illustrate H with only the first of the four relators, $a_1 = [a_1, a_2]$. There would be three similar curves relating the other consecutive pairs of holes: