Old-time Commutator Calculus

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0 Introduction

This work has its roots in the well-known these of L. C. Siebenmann [4] in which he gives necessary and sufficient conditions for an end of an open manifold to be collarable. Our generalizations have all relaxed Siebenmann's condition of stability of the fundamental group of the end. In general, we have shown that his 'inward tame' condition implies semi-stability of the fundamental group of the end and stability of the $\mathbf{Z}$-homology of the end [1].

We continue our efforts to understand the structure of an inward tame end of a high dimensional manifold. In particular, we discuss an example that exhibits minimal (in some sense) structure among all inward tame ends. As has become our habit, we use an inverse sequence of finitely presented groups and homomorphisms as a blueprint for constructing the example (see [1]). Specifically, we identify a method of comparing the derived and lower central series of a free group that allows us prove the minimality condition.

1 Inverse Sequence of Groups and Homomorphisms

For subgroups of a given group we denote commutators of X and Y by $[X, Y]$. For each positive integer i, the group G_i has the 'same' three generators a_1, a_2, a_3 and three relators $r_{i,1}, r_{i,2}, r_{i.3}$ where the relators are inductively defined as $r_{1,1} = [a_2, a_3]$, $r_{1,2} = [a_1, a_3]$, $r_{1,3} = [a_1, a_2]$ and for $i > 1$ $r_{i+1,1} = [r_{i,2}, r_{i,3}]$, $r_{i+1,2} = [r_{i,1}, r_{i,3}]$, $r_{i+1,3} = [r_{i,1}, r_{i,2}]$. We will abuse notation by not distinguishing among the various generators of the G_i's as elements of these groups.

Observe that the relators of G_i are consequences of the relators of G_{i-1} for $i > 1$. Thus, if the μ_i maps each generator a_j of G_i to the corresponding generator of G_{i-1}, then μ_i induces a group homomorphism $\mu_i : G_i \to G_{i-1}$.

Elsewhere, we build a manifold end with this inverse sequence as it pro-π_1 group. This inverse sequence of groups and homomorphisms naturally leads us to study the derived series of the free group on three generators.

Definition 1.1. Let G be a group. Define $G^0 = [G, G]$ and $G^{(k)} = [G^{(k-1)}, G^{(k-1)}]$. The sequence $\{G^{(k)} | k = 0, 1, 2, \cdots\}$ is called the *derived series for* G.

Definition 1.2. Let G be a group. Define $G_0 = [G, G]$ and $G_{(k)} = [G_{(k-1)}, G]$. The sequence $\{G_{(k)} | k = 0, 1, 2, \cdots\}$ is called the *lower central series for* G.

In a free group both the intersection of the elements of the derived series and the intersection of the elements of the lower central series are trivial. Our arguments require an understanding of the relatives 'rates of convergence' of these two sequences.

2 Representing Free Groups Among the Units of a Non-commutative Ring

The following representation dates by more than fifty years. A thorough discussion appears in [3] It turns out to give precisely the information we need.

Proposition 2.1. *Let F be a free group with basis $\{a_1, a_2, \cdots, a_n\}$. Let Π the non-commuting power series ring in indeterminates $\{x_1, x_2, \cdots, x_n\}$ with $x_j^2 = 0$ for $j = 1, 2, \cdots, n$. Then the function $\beta(a_j) = 1 + x_j$ $(j = 1, 2, \cdots, n)$ induces a representation of F into Π^*, the multiplicative group of units of Π.*

In Π, the *fundamental ideal* Δ is the kernel of the homomorphism $\rho : \Pi \to \mathbb{Z}$ that takes each x_j to 0. The elements of Δ are all sums of the form $\sum_{\nu=1}^{\infty} \pi_\nu$ where each π_ν is a monic polynomial of degree at least one. Consequently, for any positive integer k the ideal Δ^k is made of all sums of the form $\sum_{\nu=1}^{\infty} \pi_\nu$ where each π_ν is a monic polynomial of degree at least k.

The next proposition is precisely what we need to monitor the location of commutators during the construction of our example.

Proposition 2.2. *[3] Let F be a free group and $\beta : F \to \Pi$ be the repesentation given above. Let w_1 and w_2 be two elements of F such that $\beta(w_1) - 1 \in \Delta^r \subset \Pi$ and $\beta(w_2) - 1 \in \Delta^s \subset \Pi$. Then $\beta([w_1, w_2]) - 1 \in \Delta^{r+s} \subset \Pi$.*

Finally, the following lemma can be used to analyze naturally the lower central series and the derived series for the free group generated by a_1, a_2, a_3 by representing them in the non-commutative power series ring.

Lemma 2.1. *For each positive integer m let $\phi_m : F \to G_m$ be the group homomorphism induced by the identity map on the generators. Let $\psi_m : S \to S/\Delta^{2^m}$ be the natural surjection of S onto the factor ring S/Δ^{2^m}. Then, β induces a homomorphism $\overline{\beta} : G_m \to S/\Delta^{2^m}$ so that the following diagram commutes:*

$$\begin{tikzcd} F \arrow[r, "\phi_m"] \arrow[d, "\beta"] & G_m \arrow[d, "\bar{\beta}"'] \\ S \arrow[r, "\bar{\phi}_m"'] & \Delta^{2^m} \end{tikzcd}$$

3 Closing

The details of the proofs of the above will appear in a later paper.

References

[1] C. R. Guilbault and F. C. Tinsley. Manifolds with non-stable fundamental groups at infinity, ii. *Geometry & Topology*, 7:255–286, 2003.

[2] C. R. Guilbault and F. C. Tinsley. Manifolds with non-stable fundamental groups at infinity, iii. *Geometry & Topology*, 10:541–556, 2006.

[3] R. C. Lyndon and P. E. Schupp. *Combinatorial Group Theory*. Springer-Verlag, 1977.

[4] L. C. Siebenmann. The obstruction to finding a boundary for an open manifold of dimension greater than five. *Ph. D thesis*, 1965.