1 Masses

We have used calculus to make geometric calculations of lengths, areas and volumes (the full story about volumes awaits Calculus III); we have associated with these calculations such units as meters, square feet and cubic centimeters. But in the real world we can think of physical objects existing in these geometric regions, and these physical objects have what physicists call mass. The *mass* is a physical quantity measured in some such units as grams or pounds.

Typically, a physical substance is not uniformly spread over a geometric area – a wire might be heavier along part of its length, or a plate might be thicker in some regions than in others. This leads us to the concept of *density.* We can think of this as a *function* of the geometric region in which the object exists – it is actually a derivative of mass with respect to the appropriate geometric units!

Sometimes the mass of an object is uniformly spread about in the geometric region. In this case we say that the object has *uniform* density: the density function is a constant!

For an example of varying density, suppose that a wire rests on the positive x-axis, for $1 \leq x \leq 4$, where we measure x in feet. The density might then be a function like $\delta(x) = 2x$. This means that as we move to the right on the wire, it grows heavier and heavier. The *units* for this density function should be something like pounds per foot. This is really because $\delta(x) = \frac{dm}{dx}$: we take an infinitesmal slice of the wire dx, and compute its mass dm in pounds, and then the ratio gives us the (varying) density at any point along the wire. For example, $\delta(3) = 6$, which means that near $x = 3$ the wire weighs 6 pounds per foot.

We then have a corresponding equation of differentials: $dm = \delta(x)dx$. This means that to compute the total mass of the wire, we need only compute an integral:

$$\int_1^4 \delta(x)dx = \int_1^4 2xdx = 15 \text{ pounds}$$

For two-dimensional areas, density would in principle be a function of the two variables x and y – for that full story, we need Calculus III. But we can consider areas where the density might vary with respect to only one of these variables.

For example, consider the triangular region in the first quadrant bounded by $y = 2x$, the x-axis, and the line $x = 3$. Suppose further that a metal plate rests on this area, and its density varies with the distance from the y-axis, according to the function $\delta(x) = x^2$. (We could think of the plate as getting thicker and thicker, or made of material more and more like that found on Tlön.) Notice that the density of the triangle at the tip at the origin is actually 0.

To compute the area we would need the differential $dA = 2xdx$. But to compute the mass, we need the differential $dm = \delta(x)dA = x^2 \cdot 2xdx = 2x^3dx$. We obtain the following mass:

$$m = \int dm = \int \delta(x)dA = \int_0^3 2x^3dx = \frac{81}{2}$$

Notice that this calculation only works because the density of the infinitesimal vertical slice is uniform moving up and down, that is, in y.

2 Moments and Centers of Mass

We all know from our experience on a teeter-totter that a smaller person can counterbalance a heavier person either by gaining weight, or by moving further away from the fulcrum. Archimedes said that he could move the world if he could choose his lever and fulcrum! The crucial calculation is to counterbalance "moments": the *product* of a length times a mass.

Let's look at this in the wire example above: suppose that we take an infinitesmal slice dx, with mass dm. The amount of torque on the wire measured from the origin is then the distance to the origin *times* the mass of the slice: $dM = xdm$. The capital M stands for *moment* (actually, first moment, to distinguish it from the moment of inertia, a concept we won't go into). Thus, we can compute the total first moment by doing an integral of dM.

In our particular example we get the following:

$$M = \int dM = \int xdm = \int_1^4 2x^2 dx = \frac{126}{3}$$

Now comes the crucial definition: we look for a point called the *center of mass*, at which we could concentrate all the mass, and still have the same first moment! In a one-dimensional problem we are only looking for the x coordinate, and we denote that coordinate by $\bar{x}$. The equation we then get is this one (where we have suppressed the endpoints of integration):

$$\bar{x} \int dm \;=\; \int dM = \int xdm$$

The left side of this equation is a moment calculation: distance times mass. The right side of the this equation is an infinitesismal version of a moment equation.

In practice, we solve for $\bar{x}$:

$$\bar{x} = \frac{\int xdm}{\int dm}$$

In the example of the wire this gives us the following:

$$\bar{x} = \frac{(126/3)}{15} = 2.8$$

If we are computing the center of mass of a two-dimensional object (like the plate above), we will need to compute two coordinates $(\bar{x}, \bar{y})$. We will need two moments: the moment about the y-axis M_y and the moment about the x-axis M_x. If we take a vertical slice of our area (as we did in the triangular example above) we then have an infinitesimal slice of moment $dM_y = xdm$ to integrate, before we compute the x-coordinate by computing $\bar{x} = \frac{\int dM_y}{\int dm}$. We've already computed the denominator (the mass of the plate) above: it is just 81/2. The numerator is

$$\int dM_y = \int xdm = \int_0^3 x \cdot 2x^3 \int_0^3 2x^4 dx = \frac{486}{5}.$$

We can then compute $\bar{x}$ as follows:

$$\bar{x} = \frac{\frac{486}{5}}{\frac{81}{2}} = \frac{12}{5}.$$

The computation of the y-coordinate of the center of mass of our triangular plate is a good bit trickier. We'd like to take a horizontal slice when computing M_x, because we'd like a slice at a fixed distance from the x-axis (remember that the x in the subscript reminds us that we are computing a distance to the x-axis. However, in this example, the density is *not* uniform along such a horizontal slice.

Consequently, we can do a trick, and still use the vertical slice. The difference is this: the infinitesimal vertical slice has a center of mass of its own, and it is clearly in the middle of the slice. This means that the piece of moment about the x-axis can be computed by the following: $dM_x = (1/2)ydm = (1/2)(2x)dm = x \cdot 2x^3 dx = 2x^4 dx$. We can then compute the y-coordinate of the center of mass of our non-uniform plate as this:

$$\bar{y} = \frac{M_x}{m} = \frac{\int dM_x}{81/2} = \frac{\int_0^3 2x^4 dx}{81/2} = \frac{12}{5}$$

Note that the mass is the same, and we've already computed it, and so I just plugged it in! In this example, the $\bar{x}$ and $\bar{y}$ are the same, even though the triangle is taller than it is wide. This is because the triangle gets denser as we move from left to right, but stays uniform when we move up and down.