# Low Dimensional Crumpled Laminations 

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## 0 Introduction

During the 1990's Daverman and Tinsley that in high dimensions (ambient dimension larger than 6), every one-sided h-cobordism admits a special Morse function. They obtained partial results for ambient dimensions 5 and 6 but learned nothing about ambient dimension 4.

Definition 0.1. A one-sided h-cobordism is a cobordism $(W, M, N)$ with $M$ and $N$ closed manifolds and the inclusion $N \rightarrow W$ inducing a homotopy equivalence.

Theorem 0.1. Let $(W, M, N)$ be a one-sided $h$-cobordism with $M$ and $N$ closed $n$-manifolds $(n \geq 6)$. Then there exists a continuous map $p: W \rightarrow[0,1]$ with $p^{-1}(0)=M, p^{-1}(1)=N$, and $p^{-1}(t)$ a closed manifold for each $t \in(0,1)$. If $(W, M, N)$ fails to be an $h$-cobordism, then necessarily the $p^{-1}(t)$ 's are not all of the same homotopy type and at least one $p^{-1}(t)$ is wildly embedded in $W$.

Fact 0.1. Let $i: M \rightarrow W$ denote inclusion in the above setting. Then, $\operatorname{ker}\left\{\left(i_{\#}\right.\right.$ : $\left.\pi_{1}(M) \rightarrow \pi_{1}(W)\right\}$ is a perfect normal subgroup of $\pi_{1}(M)$ that is finitely generated as a normal subgroup irrespective of dimension.

Theorem 0.2. Suppose $n \geq 6, M$ is a closed manifold with $\operatorname{dim}(M)=n$ and $P<$ $\pi_{1}(M)$ is a perfect normal subgroup that is finitely generated as a normal subgroup. Then there exists a one-sided h-cobordism $(W, M, N)$ with $N \rightarrow W$ a homotopy equivalence and $\pi_{1}(N) \cong \pi_{1}(M) / P$.

## 1 Recent Developments

The initial effort combines two fronts. The first is to find specific examples in lower dimensions that were unknown to Daverman and Tinsley. In particular, the least is
known for $n=4,5$. The second is to find a one-sided h-cobordism that ( $W, M, N$ ) that does not admit a crumpled lamination. The best place to look seems to be for $n=3$ where numerous constraints are already known.

In a search for a low dimensional crumpled lamination, we have investigated a particularly straightforward example of a group (Adam's group) that abelianizes to $\mathbb{R}$ with perfect commutator subgroup. In fact, we have wondered whether this is indeed a 3-manifold group. One presentation for Adam's group is $\left\langle a, t \mid a=\left[a, a^{t}\right]\right\rangle$.

Fact 1.1. The group $G=\left\langle a, t \mid a=\left[a, a^{t}\right]\right\rangle$ is not a 3-manifold group.
Proof. This sketch was suggested by Cameron Gordon and Bob Daverman. The group $G$ abelianizes to $\mathbb{R}$ so it contains an incompressible surface. As a result, it $G$ were a 3-manifold group it would have to be residually finite (by earlier work of Haken). However, $G$ contains the Baumslag-Solitar group $K=\langle a, b \mid a=[a, b]\rangle$ as a subgroup. However, $K$ is known not to be residually finite. Residual finiteness is a hereditary property so Adam's group cannot be a 3 -manifold group.

However, we have another candidate.
Example 1.1. The group $G_{3}=\left\langle a, b, c, s, t \| c^{-1}=a^{s}, c^{-1}=b^{t}, c=[a, b], a^{-1} t=b^{s}\right\rangle$ is a 3-manifold group. In fact, the natural 2-complex arising from this presentation embeds in an orientable 3-manifold that we will call $M_{3}$.

Fact 1.2. $G_{3}$ abelianizes to $\mathbb{Z}$ with perfect commutator subgroup. Tietze transformations yield a 3-generator, 2-relator presentation: $\langle a, b, s|\left(a^{s}=[b, a], b^{s}=\left[b^{-1}, a^{-1}\right]\right\rangle$. Thus, $G_{3}$ is an HNN-extension of the free group on two generators. In particular, its commutator subgroup is a non-trivial perfect subgroup.

Conjecture 1.1. A loop in $M_{3} \times 0$ that represents $c \in G_{3}$ bounds a locally flatly embedded grope in $M_{3} \times[0,1]$. Thus, we are set up to apply high dimensional techniques.

We have also discovered prior work that should help in the negative direction.
Example 1.2. There exists a closed 3-manifold $M$ that has the homology of a lens space but is not cobordant to any closed 3-manifold that has the homology and fundamental group of a lens space. Moreover, the commutator subgroup of $\pi_{1}(M)$ is perfect.

This kind of example shows that Theorem 1.2 above is false for $n=3$.
Example 1.3. Freedman and Quinn construct one-sided $h$-cobordisms between and homology 3-sphere and $S^{3}$.

Proof. Let $H$ be a homology 3 -manifold. Then, $H \times I$ is a 4 -manifold with boundary equal to $H \times\{0,1\}$. Apply the plus construction to $H \times I$ to obtain a simply connected 4 -manifold $W$ with boundary still equal to $H \times\{0,1\}$.

Naturally, we ask whether these F-Q examples admit a crumpled lamination.

