'RELATIVE SECOND HOMOTOPY GROUPS OF THE END'

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Our goal to find an appropriate generalization of Siebenmann's theorem to the case where the fundamental group of the end is not stable. In this note we restrict ourselves to open manifolds with a single end.

Theorem 1: (Siebenmann, 1965) A one-ended open manifold, M^n $(n \geq 6)$, contains an open collar neighborhood of infinity if and only if M^n satisfies each of the following:
(1) M^n is inward tame at infinity
(2) π_1 is stable at infinity
(3) $\sigma_\infty (M^n) \in \tilde{K}_0 (\mathbb{Z} [\pi_1 (\epsilon (M^n))])$ is trivial

Guilbault and Tinsley have shown that at least one additional condition, perfect semistability of the end, is needed in a generalization:

Theorem 2: (Guilbault and Tinsley, 2003) The following conditions are sufficient for a one-ended open n-manifold $(n \geq 7)$, W^+, to be pseudocollarable at infinity:
(1) W^+ is inward tame at infinity
(2) π_1 is perfectly semistable at infinity
(3) $\sigma_\infty (W^n) = 0$ in $\mathit{invlim} \left\{ \tilde{K}_0 (\pi_1 (W \backslash K) | K^{\text{cpt}} \subset W) \right\}$
(4) π_2 is semistable at infinity
Conditions (1)-(3) are also necessary.

"π_1 semistable at infinity" mean there exists a sequence of neighborhoods of infinity such that $\bigcap U_k = \emptyset$ and such that the induced maps, $\pi_1 U_1 \xleftarrow{\lambda_2} \pi_1 U_2 \xleftarrow{\lambda_3} \pi_1 U_3 \xleftarrow{\lambda_4} \cdots$, are surjective.
"Perfectly semistable" means that in addition, the kernels of the λ_k's can be made to be perfect.
"π_2 is semistable at infinity" has a definition analagous to that π_1 semistability.

The focus of our current research is on whether a π_2 condition (such as condition (4)) is also necessary. This note discusses a possible strategy to build an example to show that a π_2 condition must be included in any list of necessary and sufficient conditions. Thus, we must identify an obstruction to pseudocollarability.

Let W^+ be a one-ended manifold with compact boundary that satisfies conditions 1-3 of Theorem 2 above. Then there exists a nice sequence of neighborhoods of the end, $\{U_i\}$, so that if $R_i = clos(U_i \backslash U_{i+1})$, then

1) $\partial U_{i+1} \to R_i$ is π_1-surjective with the π_1-kernel of $\partial U_{i+1} \to R_i$ perfect.
2) $\left(R_i, \partial U_i\right)$ has a handle decomposition with only $(n-3)$ and $(n-2)$ handles and $\pi_j\left(U_i, \partial U_i\right) = 1$ for $j \leq n-3$.
3) $\pi_{n-2}\left(R_i, \partial U_i\right)$ is generated as a $\mathbb{Z}\pi_1\left(R_i\right)$ module by a finite collection of $(n-2)$-handles of $\left(R_i, \partial U_i\right)$, $\left\{h_1^{n-2}, h_2^{n-2}, \cdots, h_s^{n-2}\right\}$.

For $1 \leq j \leq s$ let D_j^2 be the core of h_j^{n-2} viewed dually as a 2-handle. Thus, ∂D_j^2 represents an element of the π_1-kernel of $\partial U_{i+1} \to R_i$. Since this kernel is perfect, ∂D_2 also bounds a disk-with-handles in ∂U_{i+1}. Denote by s_j^2 the sphere made up of the union of D_j^2 and this disk-with-handles.

Suppose, in addition, that W^+ is **pseudocollarable**. Then a fourth, slightly more technical condition must also hold:

4) For each i there is a cobordism, $\left(T_i, \partial T_i^-, \partial T_i^+\right)$, such that $\partial T_i^+ = \partial U_{i+1}$, $\partial T_i^- \subset int\left(R_i\right)$, $\partial T_i^- \to T_i$ is a homotopy equivalence, and $Z_i = T_i \bigcup clos\left(R_{i+1} \backslash T_{i+1}\right)$ satisfies a special condition:

> For each j there is an element $\hat{K} \in H\left(\hat{\partial} Z_i^+\right)$ whose image, k, in $H_2\left(Z_i \bigcup Z_{i-1}\right)$ equals the image of $\left[s_j^2\right]$ in $H_2\left(Z_i \bigcup Z_{i-1}\right)$.

Our goal is to construct an example of an end that satisfies properties 1-3 but not property 4.

Note: Since writing this note, we have been able to show that property 4 follows from properties 1-3 and that conditions 1-3 of Theorem 2 are both necessary and suffiecient. A note will appear the the proceedings of the 2004 topology workshop.