'Perfect subgroups of HNN extensions'

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0. INTRODUCTION

This note included supporting material for Guilbault's one-hour talk summarized elsewhere in these proceedings. We supply the group theory necessary to argue that Guilbault's tame ends cannot be pseudo-collared. In particular, we show that certain groups (the interated Baumslag-Solitar groups) cannot have any non-trivial perfect subgroups. The absence of non-trivial perfect subgroups, in turn, eliminates the possibility of non-trivial homotopy equivalences.

In contrast, we include an example of a pseudo-collared end based on groups (the interated Adam's groups) that are somewhat similar to the Baumslag-Solitar groups. We close with a discussion of a homotopy theoretic approach to this construction.

I. THE GROUPS

We use the following standard notation. We let $x^g = g^{-1}xg$ or the *conjugation* of x by g, and let $[s,t] = s^{-1}t^{-1}st$ or the *commutator* of s and t. Let S be a subset of elements of a group N. We denote by $< s_1, s_2, \cdots ; N >$ the *subgroup of N generated by S* where $S = \{s_1, s_2, \cdots\}$. If N is omitted, then $< s_1, s_2, \cdots >$ is the free group generated by the characters $s_1, s_2, \cdots$. If S and N are as above, then we denote by $ncl\{s_1, s_2, \cdots ; N\}$ the *normal closure of* S *in* N or the smallest normal subgroup of N containing S.

A group N is *perfect* if it is equal to its commutator subgroup. Symbolically, $N = [N, N] = N^{(1)}$. Equivalently, N is equal to the transfinite intersection of its derived series.

The construction of the pseudo-collared end is based on the one-relator group, Adam's group:

$$A_1 = \left\langle a_1, a_0 | a_1^2 = a_1^{\left(a_1^{a_0}\right)} \right\rangle$$

We can repeat this patterm to obtain the iterated Adam's groups, A_k, $1 \leq k < \infty$:

$$\left\langle a_0, a_1, \cdots, a_k | a_1^2 = a_1^{\left(a_1^{a_0}\right)}, a_2^2 = a_2^{\left(a_2^{a_1}\right)}, \cdots, a_k^2 = a_k^{\left(a_k^{a_{k-1}}\right)} \right\rangle$$

For each $i \geq 1$, a_i is a commutator $a_i = \left[a_i, a_i^{a_{i-1}}\right]$.

Guilbault constructs the tame end that is not pseudo-collared using the simplest Baumslag-Solitar group:

$$\begin{aligned} B_1 &= \left\langle b_1, b_0 | b_1^2 = b_0^{-1} b_1 b_0 \right\rangle \\ &= \left\langle b_1, b_0 | b_1^2 = b_1^{b_0} \right\rangle \end{aligned}$$

Again, we can iterate to obtain the group, B_k:

$$\left\langle b_0, b_1, \cdots, b_k | b_1^2 = b_1^{b_0}, b_2^2 = b_2^{b_1}, \cdots, b_k^2 = b_k^{b_{k-1}} \right\rangle$$

For each $i \geq 1$, b_i is a commutator $b_i = \left[b_i, b_{i-1}\right]$.

II. HNN EXTENSIONS

The iterations above are each of a more general form. Given $L < K$ and $\psi : L \xrightarrow{1:1} K$, we define the *HNN extension* N
$\left\langle gen(K), t | rel(K), \psi(l) = t^{-1}lt \text{ for } l \in L \right\rangle$ The extension is *split* if there is a retraction $r : K \to L$. The following are well-known for HNN-extensions.

Facts:
1. K is a natural subgroup of N
2. N naturally retracts onto $< t > \cong \mathbb{Z}$, called the *free part of* N. The kernel of the retraction is $ncl\{\{K\}; N\}$.

We also use the following well-known result about the structure of subgroups of HNN extensions:

Theorem 1: (see [KS], Theorem 6) Let N be the HNN group above. If H is a subgroup of N that has trivial intersection with each of the conjugates of L, then H is the free product of a free group with the intersections of H with certain conjugates of K.

III. THE MAIN THEOREM:

First, we list some basic propositions that easily follow from the definitions:

Proposition 1: If N and H are any groups, $\phi : N \to H$ is a homomorphism, and $P < N$ is perfect, then $\phi(P)$ is perfect.

First, a similarity between A and B:

Proposition 2: For $G = A$, $G = B$, and each $k \geq 1$, there is a surjection

$$\gamma_k : G_k \xrightarrow[g_k = 1]{} G_{k-1}$$

Then, two important *distinctions* between A and B.

Proposition 3: The subgroup, $ncl\{a_1; A_1\}$, of A_1 is perfect. Moreover, the subgroup, $ncl\{a_1, ; A_k\}$ of A_k is perfect.

And, now the crucial negative result for B_j:

Theorem 2: For $j \geq 0$, the iterated Baumslag-Solitar group B_j has no non-trivial perfect subgroups.

Proof: We induct on j. The cases $(j = 0)$ and $(j = 1)$ are handled separately. For $j = 0$, B_0 is abelian. For $j = 1$, we observe that B_1 is one of the well-known Baumslag-Solitar groups for which the kernel of the map $\psi_1 : B_1 \to B_0$ is abelian. Then, we apply the argument used for the case $j = 2$.

$(j \geq 2)$ Let

$$B_j = \left\langle b_0, b_1, \cdots, b_j | b_1^2 = b_0^{-1} b_1 b_0, b_2^2 = b_1^{-1} b_2 b_1, \cdots, b_j^2 = b_{j-1}^{-1} b_j b_{j-1} \right\rangle$$

Now, B_j now can be put in the form of the HNN group. In particular, $B_j = \langle gen(K), t_1 | rel(K), R_1 \rangle$ where

$$K = \left\langle b_1, b_2, \cdots, b_j | b_2^2 = b_1^{-1} b_2 b_1, \cdots, b_j^2 = b_{j-1}^{-1} b_j b_{j-1} \right\rangle,$$

$t_1 = b_0$, $L_1 = \{b_1; B_j\}$, $\phi_1(b_1) = b_1^2$, and R_1 is given by $b_1^2 = b_0^{-1} b_1 b_0$. The base group, K, obviously is isomorphic to B_{j-1} with that isomorphism taking b_i to b_{i-1} for $1 \leq i \leq j-1$. Define $\psi_j : B_j \to B_{j-1}$ by adding the relation $b_j = 1$ to the group B_j. By inspection ψ_j is a surjective homomorphism. We assume that B_i contains no non-trivial perfect subgroups for $i \leq j-1$ and prove that B_j has this same property. To this end, let P be a perfect subgroup of B_j. Then, $\psi_j(P)$ is a perfect subgroup of B_{j-1}. By induction, $\psi_j(P) = 1$. Thus, $P \subset ker(\psi_j)$. By the inductive hypothesis, K has no perfect subgroups. Moreover, $b_1 \in K$ still has infinite order in both K (by induction) and B_j (since K embeds in B_j). Moreover, the HNN group, B_j, has the single associated cyclic subgroup, $L = < b_1; B_j >$, with conjugation relation $b_1^2 = b_0^{-1} b_1 b_0$. Recall that $\psi_j : B_j \to B_{j-1}$ is defined by adding the relation $b_j = 1$ to B_j. Thus, $ker(\psi_j) = ncl\{b_j; B_j\}$.

CLAIM: No conjugate of L non-trivially intersects $H = ncl\{b_j; B_j\}$

Proof of Claim: If the claim is false, then L itself must non-trivially intersect the *normal* subgroup, $ncl\{b_j; B_j\}$. This means that $b_1^m \in ncl\{b_j; B_j\} = ker(\psi_j)$ for some integer $m > 0$. Since $j \geq 2$, then $\psi_j(b_1^m) = \psi_j(b_1)^m = b_1^m = 1$ in B_{j-1}, ie, b_1 has finite order in B_{j-1}. This contradicts our observations above.

We continue with the proof of Theorem 2. Recall that P is a perfect subgroup of $ker(\psi_j)$. It must also enjoy the property of trivial intersection with each conjugate of L. We now apply Theorem 1 to the subgroup P to conclude that P is a free product where each factor is either free or equal to $P \cap g^{-1} K g$ for some $g \in B_j$. Now, P projects naturally onto each of these factors so each factor is perfect. However, non-trivial free groups are not perfect. Moreover, by induction, K (or equivalently $g^{-1} K g$) contains no non-trivial perfect subgroups. Thus, any subgroup, $P \cap g^{-1} K g$, is trivial. Consequently, P must be trivial. This completes the proof of Theorem 2.

IV. GEOMETRY AND HOMOTOPY THEORY

We begin this section by emphasizing the similarities between A and B. We let G stand for *either* A or B, g stand respectively for *either* a or b, and $w(g_j)$ stand respectively for *either* $g_j^{g_{j-1}}$ or g_{j-1}. Then,

$$G_k = \left\langle g_0, g_1, \cdots, g_k | g_1^2 = g_1^{w(g_1)}, g_2^2 = g_2^{w(g_2)}, \therefore, g_k^2 = g_k^{w(g_k)} \right\rangle$$

represents either A_k or B_k. For each $i \geq 1$, g_i is a commutator $g_i = \left[g_i, w(g_i)\right]$. We can summarize the relationship as follows:

SIMILARITES AND DIFFERENCES (G equals A or B)

Properties	Adam's	Baum.-Solitar
1 Relator Group	$\surd$	$\surd$
$G^{(1)} = ncl\,(g_1; G)$	$\surd$	$\surd$
Perfect $G^{(1)}$	$\surd$	$\times$
Abelian $G^{(1)}$	$\times$	$\surd$
Abelianizes to $\mathbb{Z}$	$\surd$	$\surd$
$\mathbb{Z} \xrightarrow[\text{split}]{HNN} \cdots \xrightarrow[\text{split}]{HNN} G$	$\surd$	$\surd$

Moreover, g_k is the commutator of itself with another element, $g = w(g_k)$, of G_k. So, if G_k is the fundamental group of a high-dimensional manifold, M_k, then g_k bounds a disk with one handle in M_k where one of the handle curves is homotopic (rel basepoint) to g_k. We let this be the meridianal handle curve:

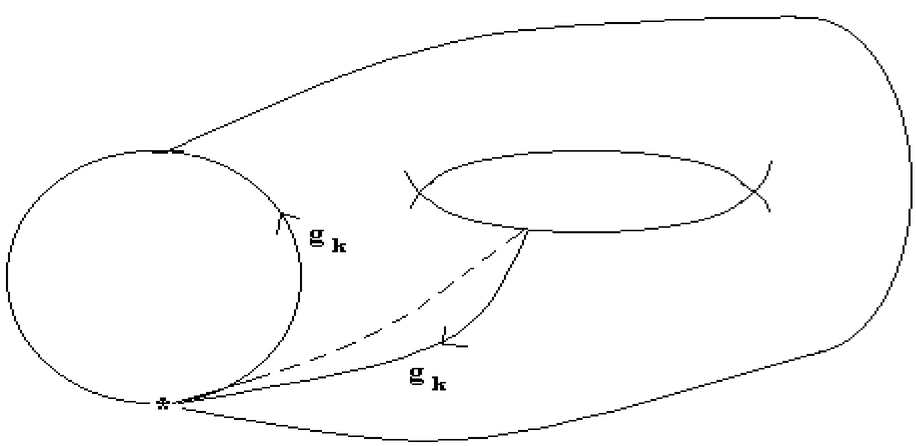

Figure 1

We can attach a two handle to M_k along g_k. Then, the disk with handle and three copies of the core of the two handle form a 2-sphere, S_k^2, along which a three handle can be attached.

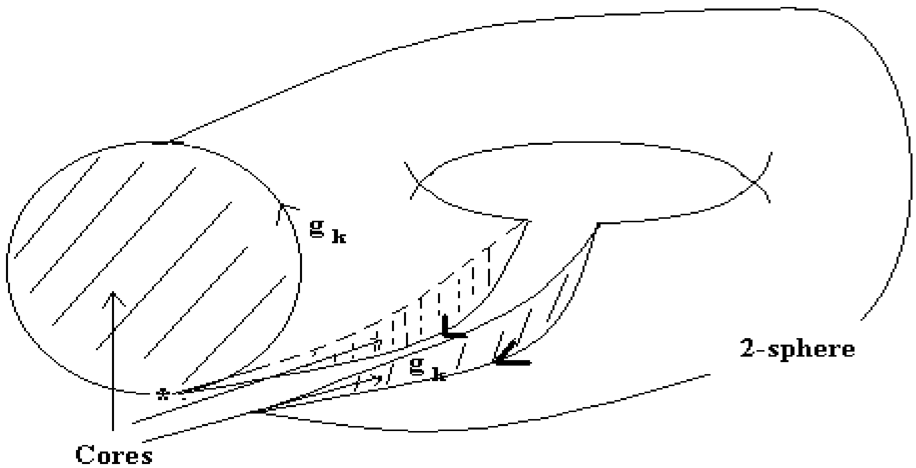

Figure 2

Note that this S_k^2 will algebraically cancel the 2-disk since it will be attached twice with one sign and once with the opposite sign. As a result the manifold, N_k, resulting from attaching these two handles to M_k, will have the same homology as M_k. In fact, we obtain a cobordism, (W_k, M_k, N_k), where the inclusion $N_k \to W_k$ induces an equivalence of homology groups.

This inclusion also induces an isomorphism on fundamental groups: $\pi_1(N_k) \xrightarrow{\cong} \pi_1(W_k) \cong G_{k-1}$. One might conclude that $N_k \to W_k$ induces a homotopy equivalence. However, this is the case only for Adams group.

The Hurewitz Theorem is needed to argue from data about homology to conclusions about homotopy. It requires simply connected spaces. Thus, we must pass to the universal covers, $\tilde{W}_k$ and $\tilde{N}_k$, of the manifolds, W_k and N_k and the cover, $\hat{M}_k$, of the manifold, M_k, that corresponds to the π_1-kernel of the induced map, $\pi_1(M_k) \to \pi_1(W_k)$.

The key becomes the longitudinal curve, g, on the disk-with-handle (shown below in bold).

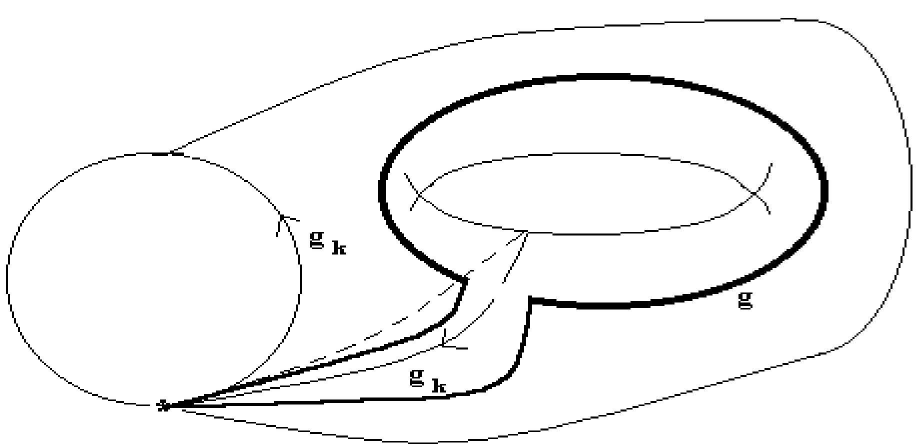

Figure 3

It is quite different for the A_k and B_k. For Adams group, $g = a_i^{a_{i-1}} = a_{i-1}^{-1} a_i a_{i-1}$ while for the Baumslag-Solitar group, $g = a_{i-1}$. In the first case, the element, $g = a_{i-1}^{-1} a_i a_{i-1}$, is a conjugate of a_i, and, in particular, lifts **as a loop** to $\hat{M}$. Consequently, the same cancellation in homology occurs in the universal cover as in the space itself and, thus, homology equivalences will yield a homotopy equivalence. In the second case, the conjugating element, b_{i-1}, lifts as an **arc** to $\hat{M}$. Thus, the two copies of the core of the 2-handle that cancelled as elements of $H_2(W, M)$ represent **distinct** generators of $H_2\left(\tilde{W}, \hat{M}\right)$ that have different signs but, in fact, do not cancel.

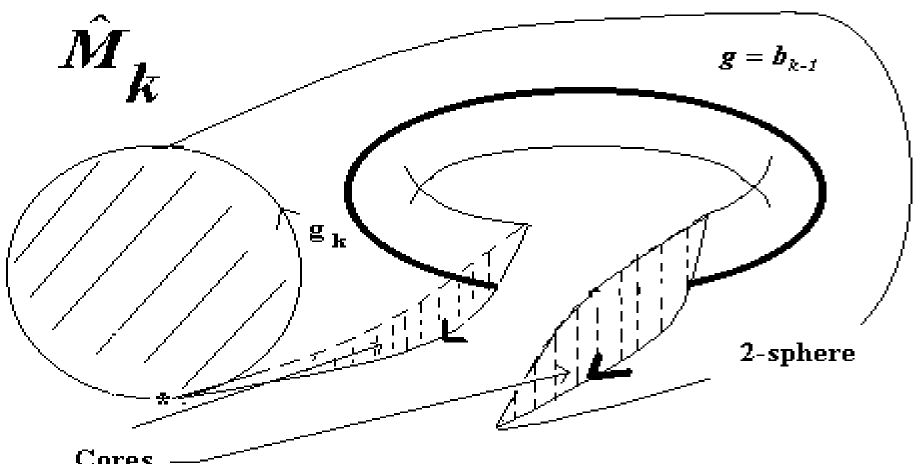

Figure 4

REFERENCES:

1. A. Karrass and D. Solitar, Subgroups of HNN groups and groups with one defining relator, Canad. J. Math. **23** (1971), 627–643.