Notes on the Schwarzschild Solution

Introduction

Karl Schwarzschild solved Einstein's equations of general relativity to determine the spacetime curvature near any mass with spherical symmetry. Planet, stars, and non-rotating black holes are all spherically symmetric. Given Schwarzschild's solution, it is relatively easy to compute the perihelion shift in the orbit of Mercury, bending of light by the sun, and gravitational time dilation[1]. Some of the results of the Schwarzschild solution are describe below.

Schwarzschild Radius

The Schwarzschild radius R_s is a characteristic radius associated with any spherically symmetric distribution of mass. Since space is curved around massive objects, calling it a "radius" is somewhat misleading. It is actually a radial coordinate that is determined by measuring the circumference C of a circle centered on the mass and dividing by 2π, $r = C/2\pi$. If any spherical distribution of mass is compressed so as to fit inside R_S, the force of gravity from the contained mass would be so great that no known force could stop the mass from continuing to collapse in volume into a point of infinite density: a gravitational singularity. This point of no return specified by R_s is sometimes called the event horizon. This special radius depends only on the mass of the object,

$$R_s = \frac{2GM}{c^2}, \tag{1}$$

where M is the mass of the object and, G is Newton's gravitational constant. R_s for the sun is 3 km so another way of expressing the Schwarzschild radius is

$$R_s = 3.0\frac{M}{M_\odot}\text{km}, \tag{2}$$

where $M_\odot$ is the mass of the sun.

Gravitational Time Dilation

Einstein showed that clocks near a massive objects would run slow. For a spherically symmetric object, one can use the Schwarzschild solution to compute exactly the time

[1]Einstein computed all of these effects before Schwarzschild developed his solution, but he used rather indirect physical arguments instead of the Schwarzschild solution.

dilation,

$$t = \frac{t'}{\sqrt{1 - \frac{R_s}{r}}}, \tag{3}$$

where t is the time measured on a clock far from the mass, t' is the time of a clock at radial coordinate r.

Gravitational Redshift

The gravitation redshift is related to the time dilation formula since the frequency of light is inversely proportional to it's period and it's period must suffer time dilation. Gravitational redshift written in term of the frequency is

$$f = f'\sqrt{1 - \frac{R_s}{r}}, \tag{4}$$

where the prime again indicates the frequency at r and the unprimed frequency is the frequency far from the mass.

Since the wavelength of light λ is related to it's frequency, $\lambda f = c$, we can write the redshift formula, Eq. 4, in terms of wavelength

$$\lambda = \frac{\lambda'}{\sqrt{1 - \frac{R_s}{r}}}. \tag{5}$$

Tidal Forces

Tidal forces are expressed as the difference in acceleration Δa between two closely separated points. The radial tidal acceleration difference is stretch with magnitude

$$\Delta a_r = \frac{2GM}{r^3}\Delta r, \tag{6}$$

where Δa_r is the difference in acceleration between two points separated by a radial distance Δr.

There is also a compressional tidal force in a direction perpendicular to the radial direction. This is also expressed as a acceleration between two closely space points

$$\Delta a_\perp = \frac{GM}{r^3}\Delta \ell. \tag{7}$$

where $\Delta\ell$ is the separation between the two points in the direction perpendicular to the radial direction.

Exercises

A neutron star similar to the one in the Crab Nebula has a radius of 15 km and a mass of $2.5M_{\odot}$,

1. Compute the Schwarzschild radius of an object with at mass equal to the mass of this neutron star.
2. How much time would elapse for a distant observer watching a clock on the surface of the neutron star tick-off one hour?
3. Suppose blue light of wavelength 475 nm is emitted from a source on the neutron star. What wavelength would be seen by a distant observer?
4. What are the tidal accelerations on a person falling feet first towards the neutron star just before the persons feet touch the surface?