FYE Block 2 Midterm

1. (10 pts)

$$\int \frac{x-4}{x^2-5x+6}dx$$

We use partial fractions:

$$\frac{x-4}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

and so we have

$$x - 4 = B(x-2) + A(x-3).$$

Evaluating at $x = 2$ gives $-2 = -A$, or $A = 2$. Evaluating at $x = 3$ gives $-1 = B$. We thus have:

$$\int \frac{x-4}{x^2-5x+6}dx = 2\int \frac{dx}{x-2} - \int \frac{dx}{x-3}dx = 2\ln(x-2) - \ln(x-3) + C$$

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2. (11pts)

$$\int \frac{x^4}{x^2+4}dx$$

We divide to get that

$$\frac{x^4}{x^2+4} = x^2 - 4 + \frac{16}{x^2+4}.$$

If we integrate this term-by-term, we get $x^3/3 - 4x + 8\tan^{-1}(x/2)$. (You might feel the need to make the trig substitution $x = 2\tan\theta$ to do the last integral.)

3. (10 pts)

$$\int x\sec^2 x dx$$

We should use integration by parts. Set $u = x$ and $dv = \sec^2 x dx$. Then $du = dx$ and $v = \tan x$. We then have

$$\int x\sec^2 x dx = x\tan x - \int \tan x dx = x\tan x - \ln(\sec x) + C.$$

4. (11 pts)

$$\int \sin^2\theta\cos^3\theta d\theta$$

We have an odd power of cosine and so this is the "easy" case: We use the fundamental trig identity to get this:

$$\int \sin^2\theta\cos^3\theta d\theta = \int \sin^2\theta(1-\sin^2\theta)\cos\theta d\theta.$$

Now the substitution $u=\sin\theta$ makes this

$$\int (u^2-u^4)du = u^3/3 - u^5/5 + C = \sin^3\theta/3 - \sin^5\theta/5 + C.$$

5. (10 pts)

$$\int \frac{1}{x^2+4x+5}dx$$

The denominator is an irreducible quadratic, and so we must complete the square. We get $x^2+4x+5 = (x^2+4x+4)+1 = (x+2)^2+1$. Consequently, this integral is $\tan^{-1}(x+2)+C$.

6. (12pts)

$$\int \sqrt{9-x^2}dx$$

We use the substitution $x = 3\sin\theta$. Then $dx = 3\cos\theta d\theta$, and so

$$\int \sqrt{9-x^2}dx = 9\int \cos^2\theta d\theta.$$

We use the appropriate trig identity to obtain

$$(9/2)\int (1+\cos(2\theta))d\theta = \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C.$$

Now $\theta = \sin^{-1}(x/3)$. But by another trig identity,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\frac{x}{3}\frac{\sqrt{9-x^2}}{3} + C.$$

Putting these together, we have the answer

$$\frac{9}{2}\sin^{-1}(x/3) + \frac{x\sqrt{9-x^2}}{2} + C.$$

7. (10 pts) Consider the lump of the parabola $y = 4x - x^2 - 3$ extending above the x-axis in the first quadrant. Rotate this lump around the y-axis. Set up a well-formed integral to compute the resulting volume of revolution.

   Notice that the parabola hits the x-axis when $0 = x^2 - 4x + 3 = (x-1)(x-3)$. That is, $x = 1$ or $x = 3$. We now use the concentric shell formula $dV = 2\pi rhdx = 2\pi xydx = 2\pi x(4x - x^2 - 3)dx$. The correct definite integral is thus

$$2\pi \int_1^3 (4x^2 - x^3 - 3x)dx.$$

8. (12 pts) Compute the area under the curve $f(x) = xe^{-x}$, for $1 \le x < \infty$. We need to compute the improper integral

$$\int_1^\infty xe^{-x}dx.$$

   But to do this, we first need to do integration by parts! Let's compute the antiderivative, using the parts $u = x$ and $dv = e^{-x}dx$. Then $du = dx$ and $v = -e^{-x}$. Then we have

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C.$$

   We then have $\lim_{b\to\infty}((-be^{-b} - e^{-b}) - (-1e^{-1} - e^{-1})) = 2/e$.

9. (4 pts) Define what it means for a set to be *finite*. A set is finite exactly if it can be placed in a one-to-one correspondence with an initial segment of the set of counting numbers.

10. (10 pts) Consider the set $\mathbb{T}$ of all *two element subsets* of the counting numbers $\mathbb{N}$. Argue that this set of subsets is *countably infinite*, by providing a procedure by which they can all be counted.

    We can systematically put all these sets in an infinite triangular array. In the first row, put all such subsets that contain the element 1; in the next row include all subsets that contain 2, but not 1. In the next row include all subsets that contain 3, but neither 1 nor 2. We get a picture something like this:

1,2	1,3	1,4	1,5	1,6	1,7	1,8	⋯
	2,3	2,4	2,5	2,6	2,7	2,8	⋯
		3,4	3,5	3,6	3,7	3,8	⋯
			4,5	4,6	4,7	4,8	⋯
				5,6	5,7	5,8	⋯
						...	...

There are a number of ways to thread through this array systematically to visit all such subsets exactly once. For example, move vertically down the columns, from left to right. This works because each of the columns has finitely many entries!